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CONSTRUCTIONS IN GEOMETRY PROBLEM SOLVING

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JAMES G. GREENO

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University of Pittsburgh

CONSTRUCTIONS IN GEOMETRY PROBLEM SOLVING

James G. Greeno
University of Pittsburgh

27 August 1979

Technical Report No. 3

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Abstract

Thinking-aloud protocols of human problem solvers working on geometry problems are presented and discussed. Protocols were obtained from six individuals working on nine different problems in which constructions were used. Nineteen protocols are presented with annotation and discussion, and other protocols are summarized. The primary purpose of the report is to provide documentary evidence relevant to a model of planning knowledge that provides an explanation of constructions (Greeno, Magone, & Chaiklin, 1979). The protocols are consistent with the general features of the model, but also show ways in which human problem solving involves processes more complex than those in the model. Illustrations of interaction between formal and informal reasoning processes are also noted.

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CONSTRUCTIONS IN GEOMETRY PROBLEM SOLVING

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This technical report has two purposes. First, it presents extensive documentation relevant to a model of the process of adding constructions to diagrams in solution of geometry problems. The model is described briefly in this report and in Greeno (1978); it is presented in detail in Greeno, Magone, and Chaiklin (1979). The second purpose of this report is to examine relationships between processes of formal and informal reasoning in several geometry problem contexts.

Problems were selected for this analysis in order to document processes in the model of constructions. Protocols on these problems are part of a data base that was collected in a project where six high school students were interviewed about once a week during the year in which they were studying geometry as a school subject. Problems presented to students were chosen to represent the material being covered in the course at the time of the interview. The nine problems analyzed in this report are all of the problems used during the year in which a proof or calculation was achieved by adding one or more lines to the diagram. (Problems in which the major task was to construct the initial diagram are not included.)

The need for a report such as this one arises from a weakness in our present methodology for studying cognitive processes. The main issue in evaluating a model of the kind proposed for constructions is whether the processes in the model are generally similar to the processes used by human problem solvers. Thinking-aloud protocols are the kind of data used most often in evaluating a model of this

kind. However, the model does not make definite predictions about the exact content of a protocol. It could be made to do that, but the additional assumptions needed to achieve complete specificity would be of minor theoretical interest. Therefore, the question of whether a protocol provides empirical support for a model must be answered by a judgment of whether the processes in the model provide a plausible interpretation of the performance shown in the protocol. This judgment is unavoidably informal, and in order to permit consensual validation of such judgments, samples of typical protocols are generally included in published articles. However, other investigators are not able to judge the extent to which the selected protocols represent the range of performance shown by a population of subjects and the extent to which the protocols not presented are also consistent with the general conclusions given by the author.

An easy solution to this problem would be to publish a complete collection of the protocols available in an investigation. This seems inefficient, since to use such raw data, investigators would be required to duplicate the efforts of the initial investigator in organizing the material and selecting relevant episodes for consideration. Furthermore, very few, if any, investigators would have the time to engage in such an examination of someone else's data. Therefore, I have opted for an intermediate presentation in this report. I have included quite a large number of protocols--many more than can be included in a published article. On the other hand, I have selected protocols and portions of protocols that I believe are especially relevant to the issues considered in this report. Those protocols that I have omitted are described briefly so that at least a general judgment can be made by readers about the nature of the performance that is not reported in detail. I also have added remarks to the protocols that are presented to facilitate identification of the parts of the protocols that seem relevant to the theoretical questions at issue.

The second purpose of the report, examination of formal and informal reasoning, is handled less systematically than the first. I have taken the opportunity of writing this report to comment on occasional features of problem solving that seem relevant to the general nature of interactions between syntactic and semantic aspects of problem solving. These observations are intended as suggestive guidelines for future research and analysis rather than as definitive documentation of any proposed hypotheses.

Theory of Constructions

In the interpretation of problems requiring constructions, their solutions are related to two general aspects of problem-solving capability. First, constructions are interpreted as consequences of the nature of problem solvers' strategic knowledge. Additions of components to diagrams are considered as procedures of pattern completion performed in order to provide prerequisites that are required for the execution of plans.

The second general aspect of problem solving involved in constructions is the interaction between semantic and syntactic reasoning processes. The diagram of a problem represents a set of components and relations that constitute a semantic model of the sequence of inferences that is given in the formal syntactic solution of the problem. A construction adds material to the semantic model for the problem and provides new components and relations about which formal inferences can be made. The basis of planning knowledge is assumed to be a set of patterns that permit specific formal inferences and calculations. Thus, the performance of a construction is generally motivated, more or less directly, by a goal that is generated in the search for a way to apply a syntactic inference rule.

The theoretical framework used in this discussion is a model called Perdix (Greeno, 1978; Greeno, Magone, & Chaiklin, 1979)

which includes a planning mechanism that proceeds in a top-down, hierarchical fashion similar to that developed by Sacerdoti (1977). The knowledge used for planning includes associations between goals and alternative plans for achieving specific goals. Each plan has one or more prerequisite features. When a goal is set, Perdix tests the situation for the presence of prerequisites of the plans that are associated with that goal. If the prerequisites of one of the plans are found, Perdix adopts that plan and works to achieve the goal using productions that constitute the detailed procedures involved in the plan that is adopted.

An example involves the goal of proving that two angles are congruent. This is associated with three plans. One plan uses triangles that contain the target angles and proves that the angles are congruent by proving that the triangles that contain them are congruent. A second plan uses relationships between angles that have the same vertex and proves that the angles are congruent if they have an appropriate spatial relationship such as being vertical angles or adjacent angles formed by perpendicular lines. A third plan uses relationships between angles whose sides are parallel and proves that the angles are congruent if they have a relationship such as being corresponding angles or alternate interior angles. When Perdix has the goal of proving that angles are congruent, it checks for the presence of global features that are prerequisites of these plans. The plan that uses triangles requires that the diagram contain triangles that have the target angles as components. The plan that uses relations involving a shared vertex requires that the target angles have the same vertex. The plan that uses parallel sides requires that the target angles have sides that are known to be parallel.

Constructions occur when Perdix finds that none of its available plans is feasible because all the tests for prerequisites fail. The procedures that produce constructions are based on patterns that contain the prerequisite features of plans. Perdix is able to identify components of a diagram that match a subset of the components of a satisfactory

pattern and perform an action that supplies a component that is missing. For example, a pattern that contains the prerequisites for proving congruence of triangles has two triangles that share a side. Perdix can identify a quadrilateral or a triangle as containing a subset of components of this pattern and add the needed line segment (the diagonal of the quadrilateral or a line from one vertex of the triangle to the opposite side).

Problem-Solving Protocols

The problems discussed in this report represent a considerable variety of patterns that students acquire during their study of geometry. Problems 1 and 2 (see Figures 1 and 9) can be solved by completing a pattern involving parallel lines with a transversal. Problems 3 and 4 are solved by adding a line that partitions a figure into two triangles with a shared side. This pattern is one that can be used to prove that components of a diagram are congruent, as corresponding parts of congruent triangles. Problems 5, 6, and 7 also are solved by completing triangles in the diagrams. In Problem 5 (see Figure 17), the construction is the base of an isosceles triangle added internally in a triangle specified to have one side longer than another. Problem 6 does not require a construction, but some subjects added a diagonal of a quadrilateral in order to use the triangle inequality. Problem 7 uses the Pythagorean theorem, and right triangles must be constructed so that their diagonals can be calculated. Problem 8's construction involves forming the central angle corresponding to an arc of a circle. The radius or radii used to form the central angle also are parts of isosceles triangles that are either formed or must be completed, depending on which case of the theorem is considered. Problem 9 involves constructions that transform a trapezoid into another figure--either a rectangle or a parallelogram.

As an aid to reference, figures and tables that present the protocol information are listed in the Table of Contents, pages vii-x.

Problem 1

The problem was the one shown in Figure 1. Students had begun to study relationships between angles with parallel sides. It is conventional in geometry that solutions to problems such as this require a series of steps, each corresponding to an inference that can be justified by a postulate or theorem. Thus, a solution consists of a numerical answer, accompanied by a proof that the answer is correct. The solution that I had in mind for the problem uses construction of a third line, parallel to AB and DE, through point C. This line divides $\angle x$ into two parts, each congruent to one of the given angles. This solution was not found by any of the subjects initially, and in fact only one of the students found a solution when the problem was presented initially. I presented additional problems to the students, and this resulted in their finding new ways to approach the problem.

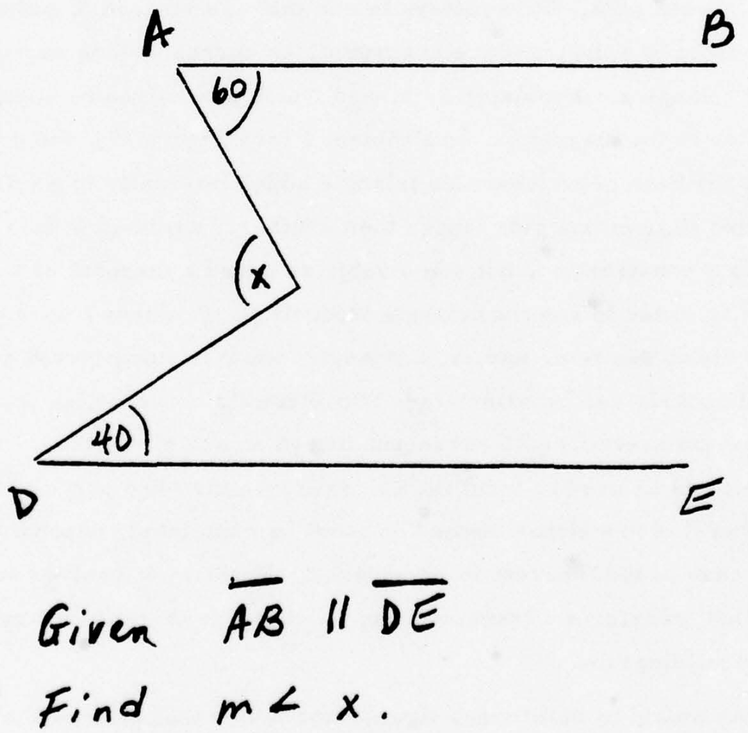


Figure 1. Problem 1.

Subject 2. The problem was solved successfully by Subject 2, whose protocol is given in Table 1.

Table 1

Protocol by Subject 2 on Problem 1

E: Now, here's a sort of an ordinary problem. (Pause.) Think out loud while you look at it and figure out what you're doing.

S: AB is parallel to DE--I'm just reading it over. (Pause.)

E: Now, when you look at the problem, tell me what you're looking at, or tell me what you're thinking about the angles . . .

S: The angles EDC and BAC, I'm thinking how they correspond to each other.

E: Mmm-hmm.

*1 S: Let me see. (Pause.) I'm wondering . . . I'm wondering about how . . . whether . . . what definition of supplementary . . . whether the three angles are supplementary or not. Let's see.

E: Why are you thinking they might be?

S: I don't know. I'm just . . . I'm drawing the conclusion for something to work from, I guess I should say.

E: Mmm-hmm.

*2 S: Let's see. (Pause.) Now, my train of thought is . . . I'm drawing auxiliary lines in my head and I'm trying to figure out where . . . I'm trying to find out what other, you know . . .

E: Mmm-hmm.

S: . . . make other things in my mind.

E: Okay. Let's draw a rough sketch of the thing down here, so you can show me what sorts of auxiliary lines you're considering. Okay?

*3 S: Okay. I was considering these two.

E: Mmm-hmm. Those are extensions of . . .

S: Yeah, of these, of A.

E: Okay.

S: Uhm . . . shoot. I don't think they're going to be much help here. Let me see. Oh, hold it. (Pause.) Okay. (Pause.)

E: What came to mind when you said hold it?

S: Oh, I was just thinking of . . . I was looking at these two angles and these two angles, and whether . . . they're not interior . . .

E: You mean, sixty and x? Is that what you were thinking of?

S: Yeah.

E: Okay.

*4 S: Now, if there's a line, a transversal, that cut them straight, then forty and sixty, then this . . . say C was on that transversal, the CDE and BAC would add up to eighty, or add up to one eighty. (Pause.) And, for some reason, I'm tending to think that C would be eighty, but I can't draw that conclusion without proof.

E: Mmm-hmm.

S: I need something more to work on.

E: Okay. (Pause.)

*5 S: All right, now, if this were sixty, then this angle must be one twenty, making this angle also sixty. All right, making this angle eighty. Now, if this was forty . . . this angle is also forty, and this is one forty, and this also is eighty.

E: Okay.

S: Now, therefore, the C must be one hundred because the two must add up, are supplementary angles.

E: Mmm-hmm. Because of the way you drew the lines . . .

S: And so angle x is one hundred.

E: Uh-huh, good, that's right. Great.

S: I'm not sure if I figured it out the right way. There must be an easier way than that.

E: I'm not sure there is. How did you know that this was going to be eighty after you knew that that was a hundred and twenty?

*6 S: Yeah. I just always know that the angles of a triangle add up to a hundred and eighty.

E: I see. Okay. So you were really using these two being a hundred . . .

S: Uh-huh.

E: . . . and subtracting that from a hundred and eighty.

S: Right.

E: Fine.

The diagram drawn by this *subject* is *shown* in Figure 2. The comment at *1 apparently indicates that the subject tried to find a relation using the information in the diagram. At *2, the subject considered additional lines, which were put into a sketch of the problem at *3. At *4, the subject considered the relation that the two given angles would have if they were formed by a single transversal. Then at *5, the subject applied that same relation to angles formed by the transversal that had been added to the diagram in the construction and used the sum of angles in a triangle (mentioned at *6) to get sufficient information to solve the problem.

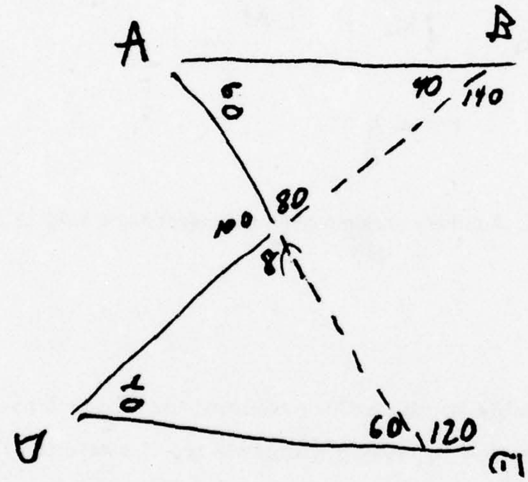


Figure 2. Drawing by Subject 2 on Problem 1.

<u>Subject 6</u>. Subject 6 did not solve the problem intially. This subject extended the lines AB and DE toward the left in the diagram and inferred the measures of the angles supplementary to those that were given, but did not add other constructions and did not see how to proceed further. I then gave Subject 6 the problem shown in Figure 3.

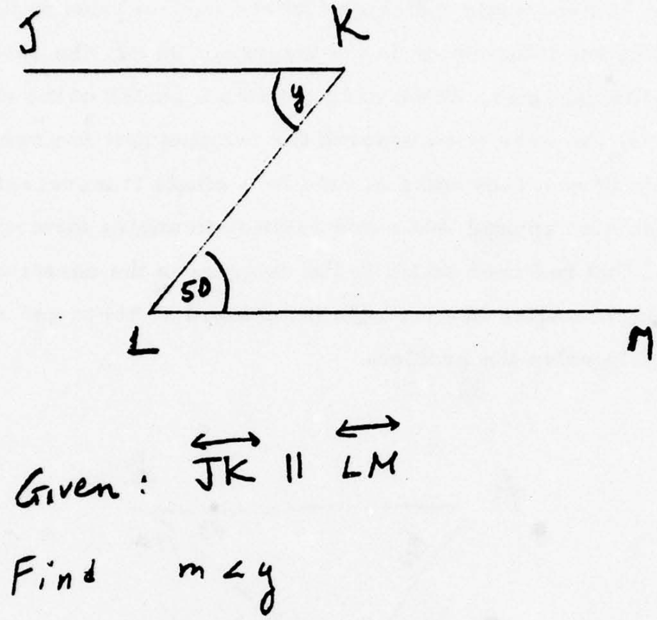


Figure 3. Auxiliary problem given to Subject 6 and Subject 5.

The subject was unable to solve the problem; the subject mentioned that the angles appeared to be complementary. I sketched the diagram with lines extended showing the usual pattern of horizontal lines with an oblique transversal and reminded the subject about alternate interior angles. The subject appeared to understand the solution.

We returned to Problem 1; the protocol is in Table 2. At *1, the subject recognized the discrepancy between the diagram and the pattern used in the intervening problem, and at *2, the subject found a way to supplement the diagram to provide transversals, as shown in Figure 4. At *3 and *4, the subject saw that inferences could be made about some of the angles in the diagram, but this did not lead to a solution. (The

fact that the sum of angles in a triangle is 180^{o} had not yet been given in class; Subject 2 had recalled that from earlier study.)

Table 2

Protocol by Subject 6 on Problem 1

	E:	Now I'd like to ask you whether that gives you any ideas about maybe solving this problem. That's forty, that's sixty, and that's x.
*1	S:	Okay. Well, if . . . okay. (Pause.) I was going to say it looks like . . . not corresponding, but . . . see, the thing in that other one is that it was cut by a transversal, and in this one it's crooked in here, and it's hard to match them up, so I . . .
	E:	Right.
*2	S:	Uhm . . . well, it almost looks like if you extended this line down to there . . .
	E:	Mmm-hmm, right.
	S:	. . . and that line up to there, you'd have a transversal.
	E:	Right.
*3	S:	And it would work either way. If you took this transversal right here, you'd have . . . let's see . . . alternate--I mean--yeah, alternate interiors.
	E:	Right.
*4	S:	And if you went with this transversal, you'd have one on either side, or alternate interiors.
	E:	Mmm-hmm.
	S:	So . . . no, wait a minute. Excuse me. They couldn't be alternate interiors, because they'd have to be equal. (Pause.) Wait a minute. Okay, I've got a transversal here . . . all right, so if I take these and just extend these that way--this will just help me, so then I can . . . and they're cut by a transversal. That means corresponding angles are going to be equal. So this would correspond with this, and this would correspond with . . . this angle would correspond with . . . I don't know. I can't figure it out.
	E:	That's okay. That's a hard problem.

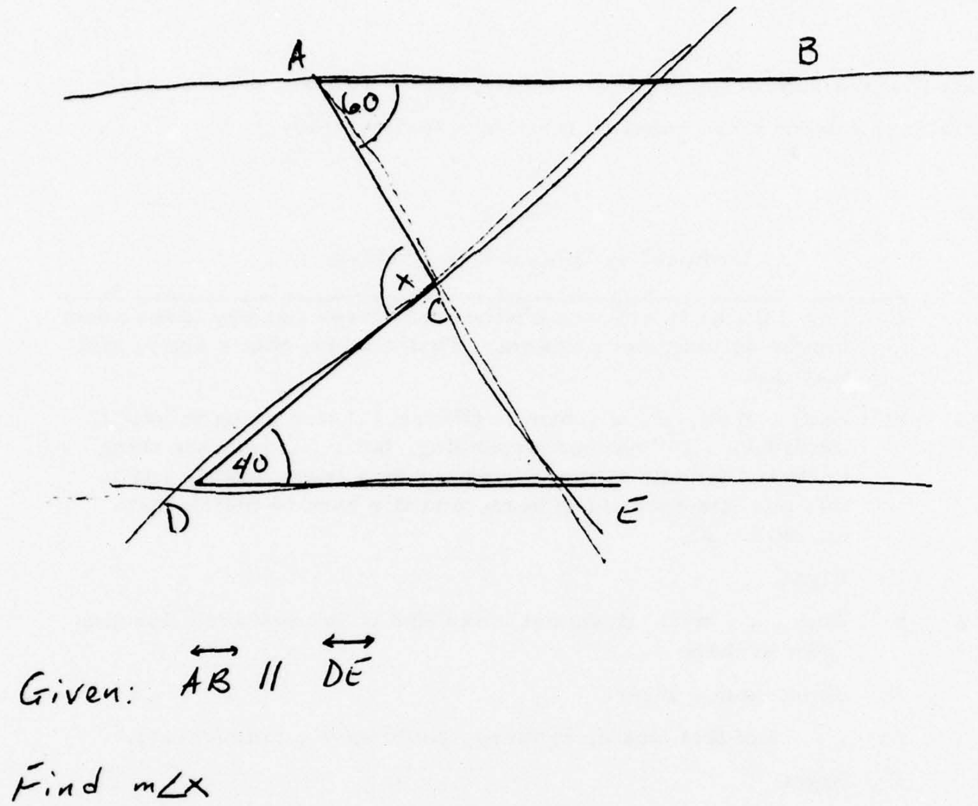


Figure 4. Drawing by Subject 6 on Problem 1.

The constructions made initially by Subject 2, and after work on the problem of Figure 3 by Subject 6, seem consistent with the idea that constructions are motivated by patterns that can be completed by adding lines to the diagram. The pattern involved in these constructions is apparently the pattern of parallel lines intersected by a transversal. The construction was apparently not motivated by a clear idea of the way in which it would lead to solution of the problem. Subject 6 did not find a solution, and Subject 2 only found the relation of angles in a triangle after considering other possibilities. Thus, the construction can best be interpreted as a result of working forward, seeing a way to modify the diagram by completing a pattern that was

partially matched in the initial situation and that seemed to provide additional relations involving the given information.

Subject 5. A different solution was found by Subject 5, who also failed to solve Problem 1 initially. This subject mentioned the possibility that the two given angles with the unknown angle might be supplementary (i.e., all add to 180°) and inferred the supplements of the given angles, as Subject 6 did. I gave the problem in Figure 3. Subject 5 gave the answer 40°, but when I mentioned alternate interior angles, the subject changed the answer to 50°. When I asked why the first answer had been 40°, the subject said, "Well, I was thinking of a perpendicular line." During work on the problem in Figure 3, the subject asked to see Problem 1 again. The subject said, "I could think of it that way with this one." I said, "What way?" and the subject said, "Put the perpendicular line." The subject then added the construction shown in Figure 5 and solved the problem using the sum of angles in a triangle. (Most of the discussion in the protocol related to arithmetic error that led the subject to the answer of 110° rather than 100°.)

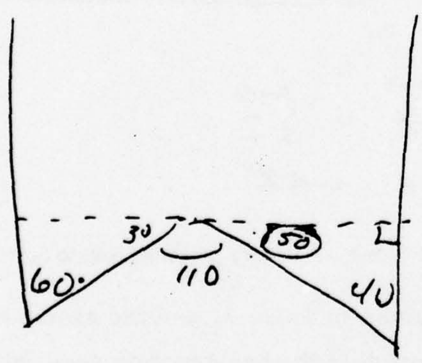


Figure 5. Drawing by Subject 5 on Problem 1.

Subject 5's solution to Problem 1 is unorthodox, but legitimate. It is unclear why Figure 3 made the subject think of perpendicular lines, but having had that idea, it provided a pattern for Problem 1 that led to a solution.

<u>Further work by Subject 2</u>. None of the constructions mentioned thus far agreed with the one I had in mind when I presented the problem. Subject 2, who solved the problem, was given the problem shown in Figure 6.

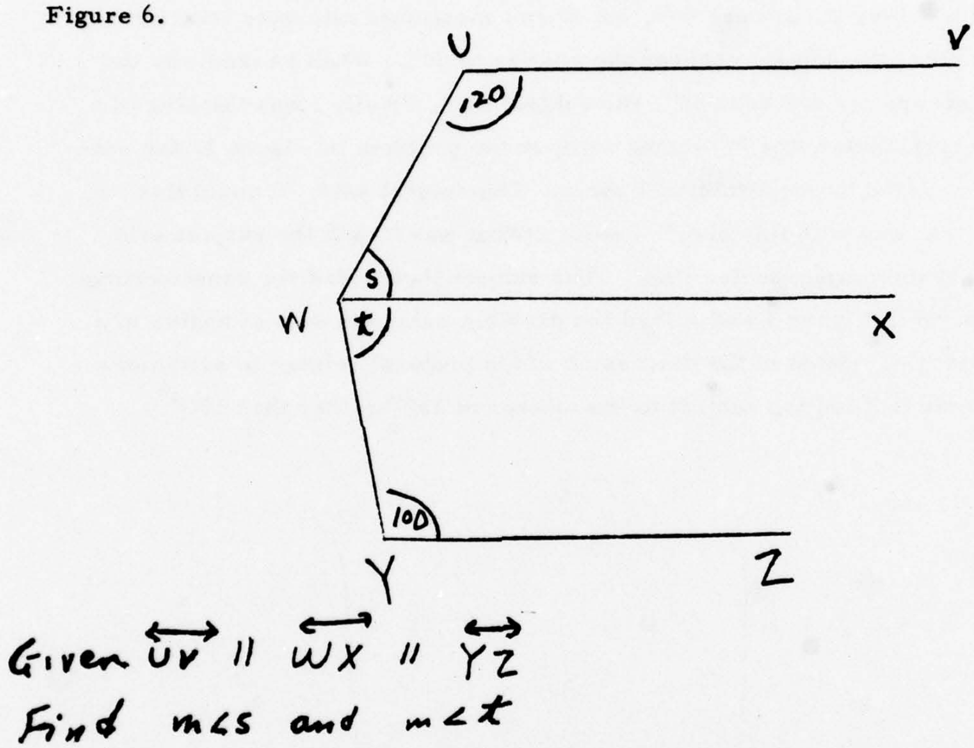


Figure 6. Auxiliary problem given to Subject 2.

The protocol is given in Table 3, and the sketch drawn by the subject is in Figure 7. Note that the construction used initially completes the pattern of parallel lines intersected by a transversal, which the subject mentioned at *1 in the protocol. At *2, the subject noticed the simpler solution based on direct relationships between the given and unknown angles.

Table 3

Protocol by Subject 2 for Problem in Figure 6

	S:	Angle s and angle t . . . all right.
	E:	Can you tell me at all what you're looking at in the diagram when you're first getting set up and thinking about the problem?
*1	S:	I'm actually thinking about the way that angle . . . or that line YW would correspond to VU and UW, how it would hit YZ and how that they would correspond to each other.
	E:	Do you want to draw a diagram?
	S:	Yeah. I think I could probably . . . I might be able to . . . figure it out the same way. I'm not feeling very mathematical about the way I'm going about this. (Pause.) Hold it. (Pause.) Oh.
	E:	What did you think of?
*2	S:	Very simple. This is eighty and this is sixty by the interior angles on the same side of a transversal add up to a hundred and eighty.
	E:	That's great.
	S:	Should have thought of that first.

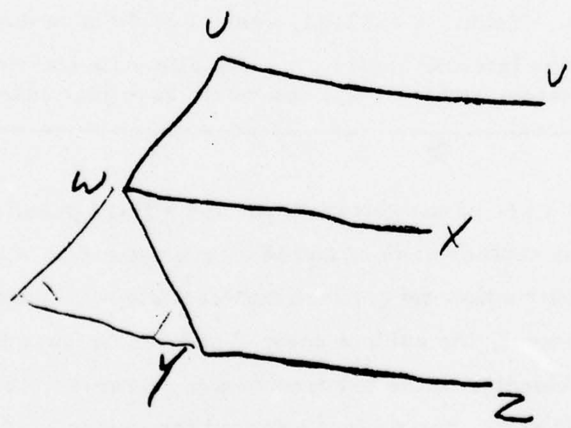


Figure 7. Drawing by Subject 2 for the problem in Figure 6.

After solving the problem in Figure 6, Subject 2 spontaneously mentioned that Problem 1 might be solved using the same pattern. The protocol is in Table 4, and the sketch drawn by the subject is in Figure 8.

Table 4

Protocol by Subject 2 on Problem 1, Repeated

	S:	I think I could have used that thing in the other one, too.
	E:	Could you?
	S:	Hold it.
	E:	You want to go back and look at it, see if there's a way? Here, you can hold onto this one if you like. Start over with another diagram if you want to.
	S:	If there's another parallel line, across, with C, if . . . if there's another parallel line here . . . and . . . hold it.
*1		I can't, because I don't know whether--I wouldn't know whether this line with C as an end point would bisect it, x.
*2		If it did I could figure it out, because I have forty here, therefore this must be one forty here. It's on the other side of the line, so this has to be forty also. Oh, yeah. I could have. (Pause.)
*3		Sixty. Yeah. A hundred, yeah, I could have done it.
*4		It's the interior angles . . . the alternate interior angles are the same, and the . . . you would have just added them up.

Subject 2's use of the pattern involving a third parallel line appeared to be yet another case of completing a pattern in working forward. The construction did not lead immediately to a solution. In fact, at *1 in the protocol, the subject seemed dubious because the added line was not a bisector of the unknown angle. However, in continuing to work forward at *2, the subject inferred the measure of part of $\angle x$, and at *3, the other part of $\angle x$ and the measure of $\angle x$ were inferred. Note that the relation of alternate interior angles was not noticed until after the problem has been solved, at *4.

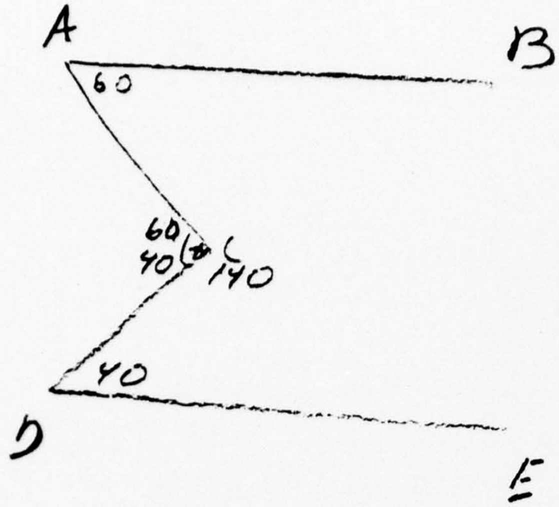
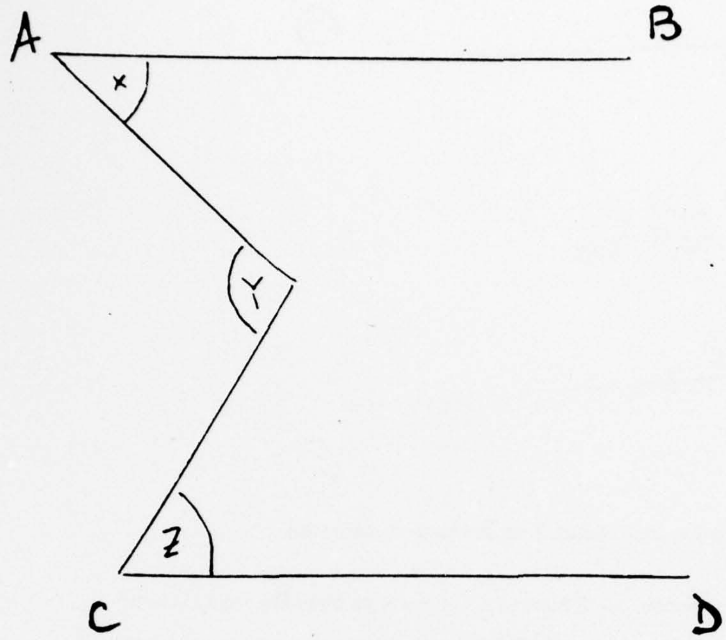


Figure 8. Drawing by Subject 2 on Problem 1, repeated.

Discussion. Protocols on Problem 1 were generally consistent with the pattern-based mechanism in Perdix for making constructions. The constructions that subjects made on Problem 1 seem to have been generated mainly to complete the pattern of parallel lines intersected by a transversal. The protocols show no evidence that any definite use of the constructions was anticipated *when they were produced.* The relationship of constructions to the search for a formal proof seems to have been rather general. Subjects apparently realized that they knew theorems that could be applied if the pattern of parallel lines and a transversal could be generated; however, they did not have definite theorems in mind.

Problem 2

The problem shown in Figure 9 was given about one week after Problem 1, and students had been working on problems involving parallel lines in the meantime. The four students who worked on this problem had successfully solved another problem that involved relationships between angles based on parallel lines, but that did not require a construction.



Given $AB \parallel CD$

Find an equation connecting x, Y, and Z.

Figure 9. Problem 2.

Subject 2. Subject 2 solved the problem successfully; the protocol is in Table 5, and the subject's work is shown in Figure 10. Note that Subject 2 constructed the third parallel line spontaneously for this problem. There is concern expressed at *1 about whether the added parallel would bisect $\angle Y$. At *2, the subject indicated that the pattern of alternate interior angles motivated the construction. At *3, the subject remarked about using variables; this led to setting up equations. The solution was apparently found at *4, based partly on examining the equations that had been written. Further discussion with the subject indicated that the subject had considered the possibility

of a bisector for $\angle Y$ because then only one variable would have been needed to find the measure of $\angle Y$.

Table 5

Protocol by Subject 2 on Problem 2

S: Find an equation connecting X, Y, and Z. (Pause.) Okay . . . okay, if I . . . if I draw a line that . . . if I draw a line that is parallel to AB . . .

E: Mmm-hmm.

S: . . . and CD, and cuts through Y . . .

E: Mmm-hmm.

S: And I have Z, and I have X.

E: Mmm-hmm.

*1 S: Okay, uhm . . . let me think . . . X, angle X is congruent to . . . could be tough. (Pause.) Oh, that won't work, because . . . my thought process isn't working because I'm thinking that Y is bisected, and I'm not sure whether it could be.

E: Hmm.

S: Uh . . . (pause) . . . hmm.

E: Why did you think of drawing the parallel?

*2 S: Because then I could work with Y, or a portion of Y being an alternate interior angle.

E: Mmm-hmm, okay.

S: And X and the portion of Y being an alternate interior angle.

E: Mmm-hmm. (Pause.)

S: Now . . . if I use . . . gosh. If I could use a variable before the Y . . .

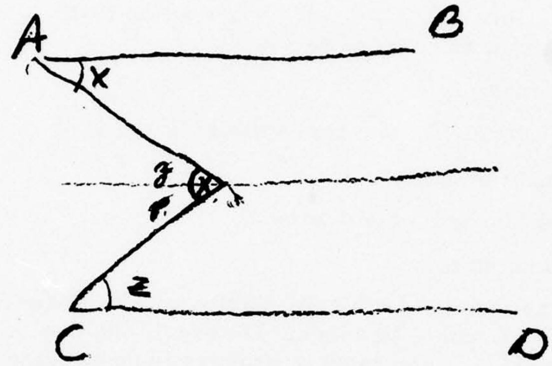
E: Mmm-hmm.

*3 S: . . . for the . . . if I could use the variable X and Y, or . . . we don't want the same letters. Z and P.

E: Fine.

S: Okay. (Pause.) Measure of angle . . . we know that measure of angle . . . Q, plus the measure of angle P equals the measure of angle Y. Okay. We know measure angle P equals measure angle Z, and we know measure angle Q equals . . . measure angle . . .

*4 Oh, okay. That's what I want. Therefore, we know that measure angle Z plus the measure of angle X is equal to measure of angle Y. Yeah, okay.



$\angle X \cong$

$m\angle \frac{1}{p} Y$

$m\angle q + m\angle p = m\angle Y$

$m\angle p = m\angle Z$

$m\angle q = m\angle X$

$m\angle Z + m\angle X = m\angle Y$

Figure 10. Drawing and writing by Subject 2 on Problem 2.

The solution given by Subject 2 differs from the one given by this subject for Problem 1 in an interesting way. In this problem, Subject 2 apparently had a rather specific idea about the use to be made of the construction, relating it to a specific theorem (alternate interior angles) rather than the general set of theorems about parallel lines. This protocol also includes an interesting example of interaction between semantic and syntactic processing. The construction was apparently motivated by a need to relate components of the problem. Once the relations were created, the subject translated them into algebraic form, and this syntactic expression apparently provided the basis for the solution.

Subject 3. Subject 3 also solved the problem successfully; the protocol is in Table 6 and Figure 11. After contemplating for a time, the subject announced the solution at *1. The protocol is retrospective.

Table 6

Protocol by Subject 3 on Problem 2

	S:	What does connecting mean? Connecting how?
	E:	It just means that there's an equation that has X, Y, and Z in it.
	S:	All right. (Pause.)
	E:	What are you thinking?
	S:	I'm not, yet, really.
	E:	What are you looking at?
	S:	The figure.
	E:	Okay. What are you trying to find there?
	S:	Uh . . . a connection.
	E:	Okay.
	S:	Sorry. (Pause.)
	E:	What kind of connection are you looking at?
*1	S:	Hang on a second. X plus Z equals Y.
	E:	That's the answer; can you show me why that works?

S: 'Cause . . . I'm going to have to draw that, wait. (Pause.) I draw a line through whatever point that is, A. That down there is Z, and that is X, and these two are now . . . well, you want me to give you theorems and postulates and all?

E: Well, just tell me a little more exactly what you mean by these two.

*2 S: Okay, for every point on the exterior of a line, you can draw one line parallel to your original line. And so, I did, and this is a transversal, and so these two are equal.

E: These two--one of them is Z--and the other one is what?

S: X and half of Y.

E: Okay.

S: So, X is equal to the other part of Y.

E: Mmm-hmm.

S: And so, Y-1 plus Y-2 equals Y, and so, using substitution you can get . . . you want me to prove this, or do you want me to just . . .

E: You're proving it.

S: No, I'm not. I mean, I'm not writing out one of those two-column proof jobs.

E: No, that's all right. You're saying everything you'd write out.

S: Yeah, okay. And then, X plus Z equals Y.

E: Okay. The only thing you didn't say is why it is that Z is congruent to that part of Y.

S: Alternate interior.

*3 E: Right. Okay, good. Now, do you remember solving problems like this before?

S: Just what do you mean by like?

E: Did you . . . solving problems where you put an extra parallel line in.

S: No.

E: Okay. (Pause.) Do you happen to know what made you think of putting an extra parallel in?

S: Well, we've done them in class, but I don't think that's what made me think of it. It's just what you do.

E: Uh-huh.

S: Well, you have to . . . to use any of the postulates you know you have to have parallel lines with transversals.

E: Mmm-hmm.

*4 S: And so the only way I was going to be able to put a connection was to have parallels with transversals.

E: Mmm-hmm.

*5 S: And so I thought about pulling those down, but that didn't get me anywhere.

E: Ah, I see. Okay.

*6 S: So I put one in the middle.

E: Mmm-hmm. Okay you did think of pulling them down, like . . .

S: Yeah.

E: Okay. And, how did you decide that that wasn't going to get you anywhere?

*7 S: There's no connection.

E: Between . . .

S: X and Z. Or X and Y, for that matter.

E: Yeah. Okay, good.

The subject made a sketch of the diagram, including two added lines. This is in Figure 11. At *2, the subject indicated the relations of congruence between $\angle x$ and $\angle z$ and the component of $\angle y$. My question at *3 was asked because this subject had solved a problem like this one about a year earlier; apparently the subject did not recall that experience. At *4, the subject indicated that the construction was motivated by a general search for connections. The remarks at *5 and *7 indicate that the subject first considered forming a transversal by extending one of the oblique lines, but that did not lead to relationships among the angles in the problem. The comment at *6 suggests that the construction that worked was probably motivated by a general search for relationships rather than by any specific theorem. (Subject 3 was not interviewed on Problem 1.)

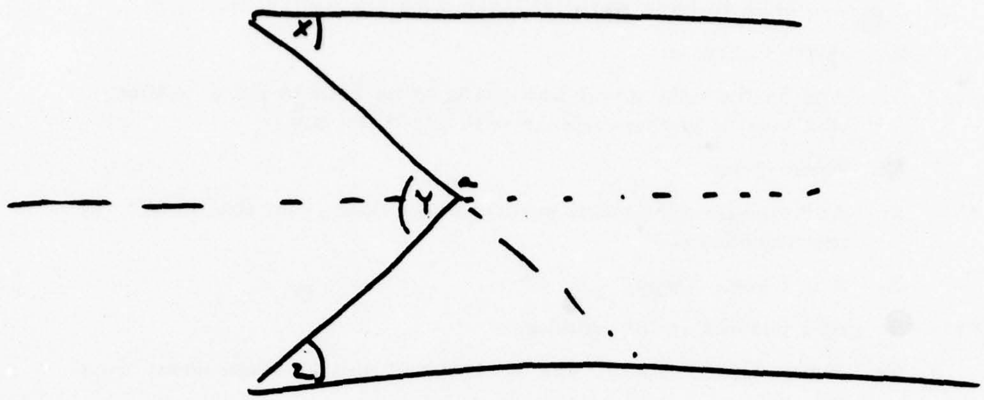


Figure 11. Drawing by Subject 3 on Problem 2.

Subject 5. Subject 5's protocol on Problem 2 is in Table 7, and the diagram drawn by the subject is in Figure 12. The subject remembered Problem 1; however, recall that Subject 5's solution to Problem 1 involved drawing a perpendicular line between the parallels, and that pattern was not used in Problem 2.

Table 7

Protocol by Subject 5 on Problem 2

E:	Now, this one might be a little tough. Have a look at this and see . . .
S:	I remember this.
E:	Oh, you do, huh? You remember it from when?
S:	One like this a couple of times . . .
E:	That's when I had one like that.
S:	Yeah.
E:	I was wondering if you'd remember it.
S:	Connecting X, Y, and Z.

E: Draw a diagram if you want.

S: Okay. I never looked this one up.

E: Good. I can't remember whether we got a solution or not.

S: Well, I think you showed me.

E: Oh, did I?

S: Well, I'm not sure.

E: Well . . . (pause).

S: Well . . .

E: What are you thinking, what are you trying to find?

*1 S: I was thinking a hundred and eighty minus X plus Z, the quantity of X plus Z, equals Y.

E: Okay. Why do you think that might be true?

*2 S: Well, because . . . that can't be right, because the larger these get, the larger this one gets.

E: Oh, that's interesting.

S: Can't be right. So, another equation.

E: How could you tell that they sort of go together that way?

S: Well, because the larger these two angles are, then . . . ultimately they'll be perpendicular to these two lines.

E: That's true. Okay.

S: And these would be ninety degrees, and this would be a hundred and eighty.

E: Okay. (Pause.)

*3 S: Well, then the other one would be X plus Z equals Y. That would be another way to do it. And X plus Z equals Y equals a hundred and eighty. (Pause.) I should remember that one. What you told me. I'm pretty sure we worked this--it might have been a different one.

E: Well, in fact it wasn't quite the same. Why don't you draw the diagram over there. And . . . and then, see what you know that you might be able to use. You may have to add something to the diagram in order to make it work.

*4 S: Well, I could extend this. I could extend the lines. I could try working it out like in a proof. I could extend this one.

E: Okay.

S: And, since these two are parallel . . .

E: Yeah.

S: And I can extend this one. (Pause.)

E: Now. Did you have any reason for extending those? What did you think it was?

*5 S: Well, it might help me, but, since these two are parallel I can look for something that I could use that will help me. One of these corollaries.

*6 S (cont'd): I'm thinking maybe I can use a triangle here.

E: Okay. (Pause.)

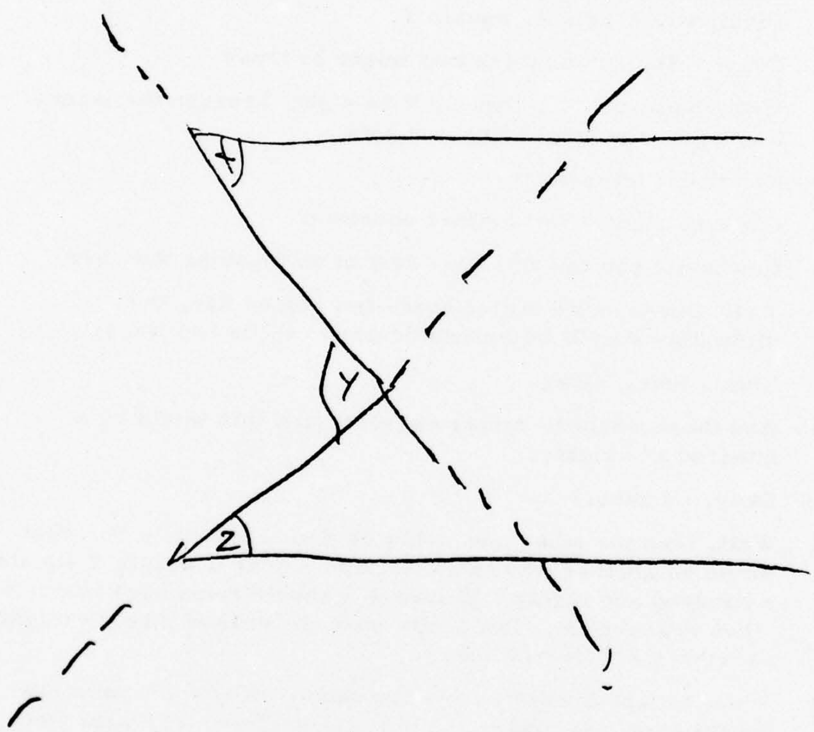


Figure 12. Drawing by Subject 5 on Problem 2.

At *1, Subject 5 mentioned a conjecture, that the three angles would sum to 180°. This conjecture also occurred to this subject in Problem 1. The comment at *2 represents an interesting instance of

semantic reasoning to test the conjecture. The subject realized that if the three angles sum to a constant, then as two of them increase the other must decrease, and that does not hold in this situation. At *3, the subject actually found the solution for a special case of the problem, where $\angle x$ and $\angle z$ equal 90° and $\angle y$ is 180°. This idea was not pursued to a general solution.

At *4, the subject formed constructions by extending the oblique lines. The comment at *5 suggests that this was done in order to complete the pattern of parallel lines intersected by a transversal. At *6, the subject mentioned that triangles are available; it is unclear whether that idea was present when the constructions were formed--the triangles might have been noticed after the construction was completed. In any case, the subject did not succeed in finding relationships that could be used to solve the problem. I suggested the construction involving the third parallel, and Subject 5 successfully solved the problem with that.

Subject 6. Subject 6 also failed to solve the problem. The construction shown in Figure 13 was drawn, with the remark, ". . . these two lines are parallel and I cut it by a transversal." Later the subject said, "I might be able to use some of my . . . some of my, you know, postulates or theorems that I know . . . to . . . to try to get an equation." However, the solution was not found. Note that this subject did not mention use of the triangle. (Recall that the class had not included study of properties of triangles when I presented this problem.) When I suggested the construction with the third parallel, Subject 6 considered the possibility that the new line bisected $\angle y$, and also that the two parts of $\angle y$ might be complementary. (Apparently, both of these conjectures were based on the appearance of the diagram.) However, the solution of the problem was found only with considerable coaching.

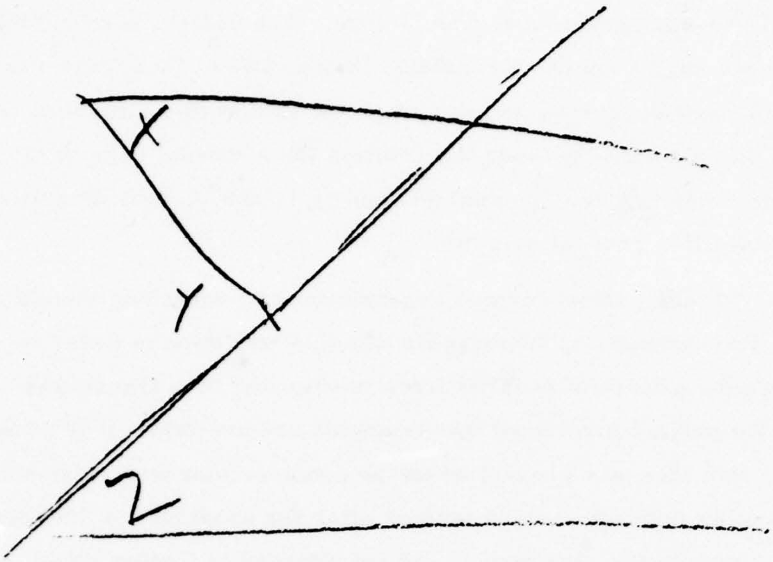


Figure 13. Drawing by Subject 6 on Problem 2.

Discussion. The protocols on Problem 2 also were generally supportive of the ideas in Perdix. These protocols gave further evidence of the use of the pattern of parallels and transversal as a basis for constructions. The pattern involving a third parallel line also was used by two students. In three of these four protocols, the constructions were apparently produced during a general search for relationships among problem components. However, Subject 2 apparently saw the use of the construction more specifically. An intriguing conjecture is that increased knowledge about the domain, including more practice in solving problems, could produce a more differentiated knowledge structure that would enable a problem solver to identify more specific patterns that would be produced by constructions. This kind of mechanism would relate closely to the well-known pattern-recognition skills of Go and chess players (Chase & Simon,

1973; Reitman, 1976) and would provide a basis for explaining the usefulness of this skill in actual play, since this requires the ability to see patterns that can be produced by appropriately chosen moves, in addition to patterns that are present in the situation.

Some examples of semantic processing also occurred in the protocols for Problem 2. The spatial processing exhibited by Subject 5 provided an interesting example of semantic testing of a conjecture, and the use of algebra by Subject 2 gave an interesting case in which semantic processing suggests a representation in the formal language.

Problem 3

In this problem, given about a month after the first two problems, students were asked to prove a theorem--the base angles theorem for isosceles triangles. In presenting the problem, I mentioned that I wanted to see whether the student might remember the proof of the theorem or figure it out. I then stated the theorem in the form: "If two sides of a triangle are congruent, then the angles opposite those sides are congruent." This theorem had been presented and used in homework, but the proof had not been memorized, and it is clear that the students were not simply recalling the proof.

Subject 2. Subject 2's protocol is in Table 8 and Figure 14. The auxiliary line was drawn at *1, and at *2 the subject mentioned that it would be used to provide congruent angles. In retrospect, at *3 the subject noted that the side-angle-side pattern had been in mind when the construction was made.

Table 8

Protocol by Subject 2 on Problem 3

S: Okay, my first step would be . . . AC is congruent to BC, and that would be given.

E: Mmm-hmm. Right.

*1 S: All right. Two would be . . . let's see, what do I want? Draw a line that bisects C.

E: Now do you have a reason for drawing it that way that you have in mind?

S: You mean instead of drawing it so that it bisects . . .

E: Yeah.

S: Why I said it bisects C?

E: Yeah.

*2 S: Because I want ACX, I'm going to call it, to be congruent to BCX.

E: Okay. (Pause.)

S: And that's . . . through an angle, a bisector can be drawn.

E: Okay.

S: Or, through an angle, only one . . .

E: Right. (Pause.)

S: And . . . three, let's see . . . okay, ACX, angle ACX is congruent to angle BCX, because definition of a bisected angle.

E: Mmm-hmm.

S: Four . . . CX is congruent to CX by the reflexive property . . . of congruent segments.

E: Mmm-hmm.

S: And finally--or not finally--triangle ACX is congruent to BCX by the side-angle-side.

E: Mmm-hmm.

S: And six . . . angle A is congruent to angle B because corresponding parts of congruent triangles are congruent.

E: Right. Okay. Now, when you decided to draw the bisector . . .

S: Mmm-hmm.

E: You said you wanted those angles congruent. Did you already have in mind that you'd use side-angle-side, or did that come up later when you saw where the line went?

S: Well, I looked at it because the restrictions that I was going to place on the auxiliary line . . .

E: Yeah.

S: I had to decide whether I had to say . . . make the line perpendicular to AB . . .

E: Right.

S: Or I could say make it bisect AB, or I could make it . . . congruent to, or make it bisect angle C.

S: And if I made it . . . let me think, I could do it two different ways. And, I looked at it and I ruled out having it bisect AB right immediately because I could see it gave me side-side.

E: Mmm-hmm. (Pause.)

S: Oh, I think all of them would work, in fact. (Pause.) Yeah, I think I could have done all of them. Because with the others you could use either side-side-side or with the perpendicular you could use the . . . you could use the HL.

E: Right.

*3 S: But when I looked at it, I didn't think of these immediately, while I saw the side-angle-side immediately, so I decided to bisect . . . I just used the one I saw first.

E: All right. So when you decided you wanted that angle . . .

S: Mmm-hmm.

E: . . . did you also have in mind that you'd be able to use the common side? Had that occurred to you at that point?

S: That just went right along with it, yeah.

E: Okay.

S: I mean, I just . . . didn't even think about that. That's something that just comes, you know, you don't even have to think in using it with triangles like that, because it's just something that's apparent.

E: Okay, great.

---

1. $\overline{AC} \cong \overline{BC}$	1. Given
2. Draw a line that bisects $\angle C$	2. Through an $\angle$ only one bisector can be drawn
3. $\angle ACX \cong \angle BCX$	3. Def. of a bisected $\angle$
4. $\overline{CX} \cong \overline{CX}$	4. Reflexive Prop. of $\cong$ Seg.
5. $\triangle ACX \cong \triangle BCX$	5. SAS
6. $\angle A \cong \angle B$	6. CPCTC

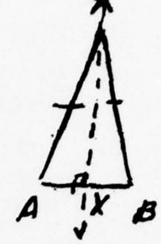


Figure 14. Drawing and writing by Subject 2 on Problem 3.

<u>Subject 3</u>. Subject 3 also solved this problem by constructing the bisector of the angle at the apex and obtaining the pattern side-angle-side. However, Subject 3's protocol was not clear about the sequence of ideas that preceded construction of the bisector. The protocol includes the remark: "I'm trying to remember how we did it in class. I was trying to remember if I put that in, how I'd either prove that this is, you know, going at right angles, or . . . oh, I know. Okay. So you draw in your auxiliary line, which is CX and . . . Shoot, I was wrong. It doesn't matter, 'cause I was wrong." The subject then gave the proof in which CX was specified as the angle bisector. It seems likely that the construction preceded clear knowledge about the relations that would be used in the proof, and that a pattern of relations was considered in deciding how to specify the auxiliary line.

Subject 4. Subject 4's protocol is in Table 9 and Figure 15. The auxiliary line was constructed at *1, where the subject remarked that its use was not yet identified. At *2, the subject had identified the auxiliary line as the angle bisector; *3, in retrospect, remarked that the pattern of congruent triangles had been in mind when the auxiliary line was drawn, and at *4, noted that the property of a shared side was a goal of the construction.

Table 9

Protocol by Subject 4 on Problem 3

	S:	Okay, if two sides of a triangle are congruent, so . . . draw a triangle.
	E:	Okay.
	S:	Then the angles opposite those sides are congruent. Okay, so, like, if I have . . . given: triangle ABC--I'll letter it ABC.
	E:	Right.
	S:	And then I have . . . prove: . . . do I already have these two sides given? Okay. Two sides of a triangle are given.
	E:	Mmm-hmm.
	S:	Let me go back to my given and say that segment AC is congruent to segment BC.
	E:	Okay.
	S:	And I want to prove that angle A is congruent to angle B.
	E:	Good.
	S:	All right. Let me write down my given. Okay. And mark my congruent sides. Okay, so, I want to prove that angle A is congruent to angle B. Now, let's see. Do you want . . . ?
	E:	Yeah. Why are you drawing a line there?
*1	S:	I don't know yet.
	E:	Oh, that's okay. Don't erase it.
	S:	I'm going to do it, no, I just . . .
	E:	Oh, okay, fine.

S: Okay . . . okay, then I could . . . if I drew a line . . .

E: Mmm-hmm.

*2 S: That would be the bisector of angle ACB, and that would give me . . . those congruent angles . . . no. (Pause.) Yeah, well, that would give me those congruent angles, but I could have the reflexive property, so this would be equal to that. Okay, I've got it.

E: Okay.

S: Okay.

E: Now, before you go ahead and write it all down, when you said you were going to draw the line . . .

S: Yeah.

E: And I said why are you doing that, and you said you didn't know yet, what do you think happened to give you the idea of making it the bisector?

*3 S: Okay, well, I have to try to get this . . . I have to try to get triangle ACD congruent to BCD. Because, if I do that, then angle A is congruent to angle B because corresponding parts of congruent triangles are congruent.

E: So you were drawing the line to give yourself triangles, is that the idea?

*4 S: No, to . . . to get a side that was in both triangles.

E: Okay.

S: And to get congruent angles.

E: So that's why you drew it as the bisector.

S: Yeah.

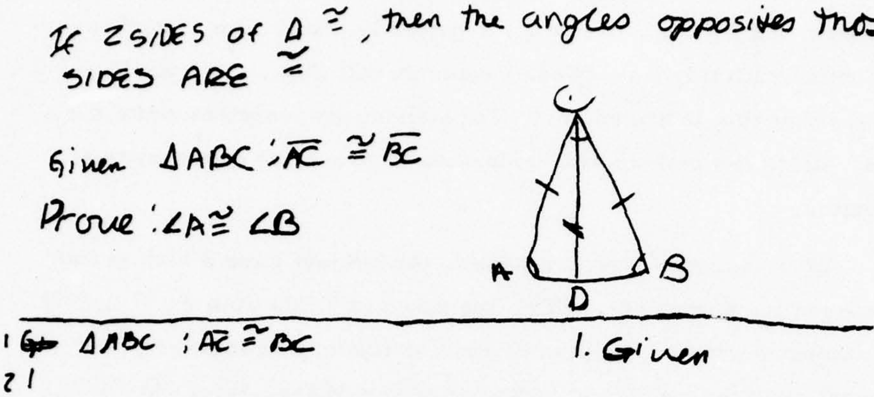


Figure 15. Drawing and writing by Subject 4 on Problem 3.

Subject 5. Subject 5 began by drawing a diagram of the triangle with two sides congruent, then said, "I want to prove these angles congruent. (Pause.) I can draw a bisector. (Pause.) And this . . . oh, that's easy." When I asked later what the subject was thinking about when the line was added, the subject said, "Well, I have to divide it up into two triangles to prove congruence, and then I could find the two sides." I said, "You thought of doing it by having the angle bisect it. Were you already thinking about getting side-angle-side at that point?" Subject 5 said, "Yeah, I was getting the two angles here congruent." The subject then went on to notice that the construction could also have been specified as the median or the altitude drawn from the apex of the triangle.

Subject 6. Subject 6 initially recalled the theorem in terms of isosceles triangles and noted that a direct proof was possible. I

explained that I was interested in having the subject prove the theorem. The subject drew the auxiliary line in the triangle, saying, "What I do is I draw a line in here. (Pause.) That's not too good a line. I'd put one here, I'd put . . . drawn . . . probably, now, like, I can't overdetermine with this line. See, I couldn't call this . . . I would probably say that this is a median." The subject proceeded to write out a proof, using the pattern side-side-side to prove the congruence of triangles.

After the proof was completed, the subject gave a rich set of retrospective comments, which are given in Table 10. At *1 and *2, the subject identified the plan of proving triangles congruent, and at *3 mentioned the pattern of triangles with a shared side. At *4, the subject mentioned the need to find one of the patterns that are sufficient to prove that triangles are congruent. At *5 and *6, there are comments about considering components of the figure. The subject apparently considered the apex angle of the original triangle, which was divided by the auxiliary line and hence was not usable. The consideration of the auxiliary line as the angle bisector would have given a different solution of the problem, but the subject apparently did not think of this during initial solution of the problem. At *6 and *7, the subject reported that the choice of the line as the median was related to a subgoal of finding congruent segments in the triangles. However, when I asked at *8 whether the goal was in mind when the decision was made to specify the line as the median, the comments at *9, *10, and *11 indicated that the specification may have been a relatively arbitrary choice, and this suggests the conjecture that the solution was more a working-forward process in which some reasonable construction was made first and was later found to be an adequate basis for a solution.

Table 10

Retrospection by Subject 6 about Problem 3

E: Now, let me ask you a little bit . . . when you first started, you decided to put a line down through there.

S: Yeah.

E: What were you thinking about when you . . .

S: Well, because I knew, when I first saw this . . .

E: Yeah.

*1 S: . . . the only way, because you didn't give me anything in the given, that talked about these angles, the only way that I was going to be able to prove them . . .

E: Mmm-hmm.

*2 S: . . . if they could be corresponding parts of two triangles.

E: Okay.

*3 S: That's the only way I could get two triangles out of it, split right down the middle. And also doing that I get a common side, of two triangles.

E: Okay, great. Then when did you think about using the sides of . . . I'm sorry, when did you think about using the segments of that bottom side of the triangle?

*4 S: Well, as soon as I drew the line, the first thing . . . the first thing I did after I put the given--I usually do this with all my proofs--

S (cont'd): is look for the common side. So once I put the common side down, then I looked for a way that I could use an angle--or a triangle congruence postulate to figure it out.

E: Mmm-hmm.

*5 S: And I looked and I had these two common sides, then I looked up here and I couldn't say that . . . I couldn't use the common angle because that's . . . because that has a split line, so that angle I have to throw out unless I made this an angle bisector. This is the only . . . I had to find another side . . .

E: Mmm-hmm.

*6 S: But . . . so the only two angles I'd have left to work with are these two, and I have to prove that, so the only thing I can do is use lines.

E: Okay.

*7 S: So, I used that.

E: Okay.

S: I could have drawn an angle bisector and used ASA, though, or SAS.

E: Uh-huh.

S: Or a number of things.

E: Right.

S: The only thing I couldn't have used here is an altitude. Because that would have been overdetermining.

E: Okay. Now, when you made it the median . . .

S: Yeah.

*8 E: . . . were you thinking at that point of being able to use the sides of the triangle, or did that occur to you later on?

*9 S: Well, that occurred to me later on.

E: Okay.

*10 S: I just . . . I realized later on, I *forgot* that a median would bisect. All I wanted was just a name for a line that was going to give me two triangles.

E: I see.

*11 S: Then I realized later I forgot that the median would split that bottom segment.

E: Okay, great.

Discussion. Two components are needed for the construction in this problem: the auxiliary line and its specification. The specification is required because the line that is added is not determined. In Problem 4, for example, the auxiliary line connects two points in the diagram. In Problems 1 and 2, the added lines either extended lines already in the diagram or were constructed with the goal of being parallel to given lines.

The protocols provide a range of processes for constructing the needed components. Subjects 2 and 5 apparently specified the construction immediately, which is consistent with performance by Perdix. This also is consistent with Gelernter's (1963) system in which

constructions are selected on the basis of some construction theorems that permit lines to be added to diagrams. Subjects 3, 4, and 6 appear to have had two phases of construction, in which a line is added to provide triangles in the diagram, and a specification is chosen later. The protocols for Subjects 3 and 4 suggest that the specification was related to a general goal of having some congruent components that were later incorporated into a solution for congruent triangles. Subject 6's retrospections suggest the alternative possibility that the construction was specified by a permissive theorem--that the median is something you can construct--and that this then guided a search for a solution. Thus, use of a construction theorem can either be associated with a goal (as it may have been for Subjects 2 and 5) or chosen in a way that relates to a pattern (Subject 6), and a construction may be drawn and left unspecified until a specification is found that satisfies a relevant subgoal (Subjects 3 and 4).

Problem 4

This problem was given about one week following Problem 3. Students were asked to prove the following theorem: "If two sides of a quadrilateral are congruent and parallel, then opposite angles of the quadrilateral are congruent." It is of interest that in the previous session, a special case of this problem was given, with a diagram containing the diagonal and a specific pair of angles to prove congruent (not the opposite angles of the parallelogram). Subjects 2 and 3 solved the previous problem, using alternate interior angles as needed. Subject 5 found a solution that was incorrect; the alternate interior angles in the problem were found, but the subject used the angles that were to be proven in an angle-side-angle pattern rather than using the diagonal as a shared side. Subject 6 was the most interesting. This subject *first suggested* incorrectly that the target angles were congruent as alternate interior angles. When I pointed out that this was not correct,

the subject adopted the plan of proving triangles congruent, but did not find the alternate interior angles that completed a side-angle-side pattern.

In Problem 4, Subjects 2, 3, and 5 all solved the problem, making an appropriate diagram and adding the diagonal as an auxiliary line. In each case, the construction was apparently motivated by the plan of proving triangles congruent. The specific components to be used in the proof were apparently discovered subsequent to constructing the diagonal. Subject 5's protocol is in Table 11 and Figure 16.

Table 11

Protocol by Subject 5 on Problem 4

S:	Well, you have a quadrilateral . . .
E:	Yeah.
S:	And . . . all right, I have two sides parallel and congruent, right?
E:	Mmm-hmm.
S:	And so I want to prove opposite angles congruent?
E:	Mmm-hmm.
S:	So I've got a line . . .
E:	Okay.
S:	And I'll put this . . . no, I want to prove this. This congruent to this.
E:	Okay.
S:	All right, so I . . . (pause) . . . these parallel lines . . .
E:	Mmm-hmm.
S:	All right, so these are given. That's the definition of a quadrilateral. These are also, but I'm trying to prove the theorem that says the opposite angles.
E:	Right.
S:	Okay, so, I have . . . I'll mark angles.
E:	Mmm-hmm.
S:	One, and two.

E: Mmm-hmm.

S: I'll prove these congruent by . . .

E: Okay.

S: . . . alternate interior angles.

E: Okay.

S: Oh, I won't write the whole thing out. (Pause.)

E: Okay, great.

S: And then I have side-angle-side to prove that.

E: Right. That's great.

S: And then I have corresponding parts.

E: Right.

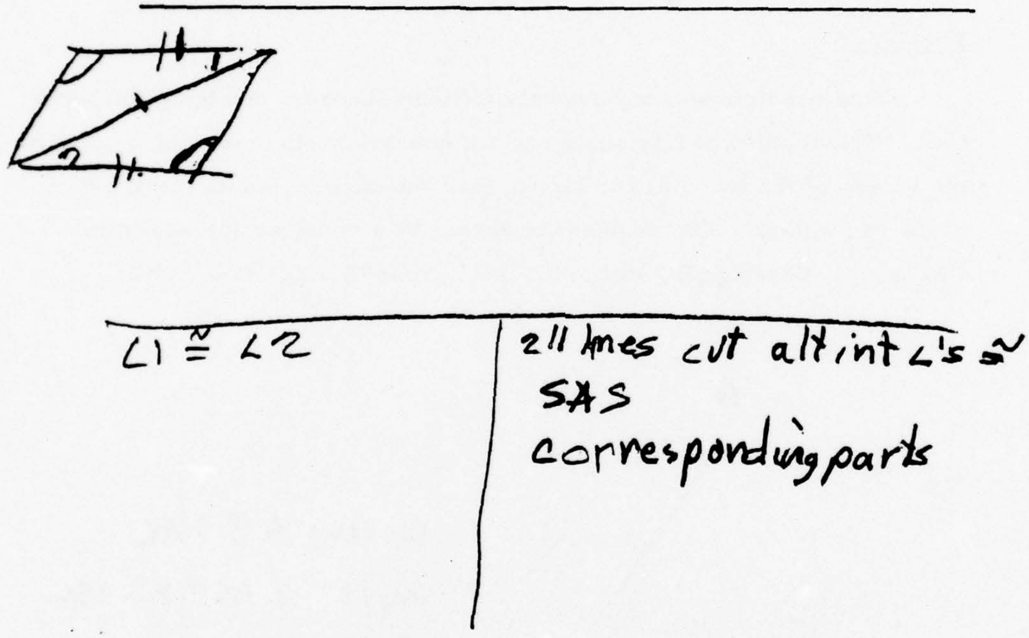


Figure 16. Drawing and writing by Subject 5 on Problem 4.

Subject 6 did not solve the problem. This subject drew a correct diagram, but did not add the diagonal. The subject extended the sides to form angles exterior to the quadrilateral and noted several relations

between angles, including interior angles on the same side of the transversal and corresponding angles. The subject noted that if the figure had been specified as a parallelogram, or if both pairs of sides were given as parallel, the problem would have been solvable.

Discussion. This problem is simpler than Problem 3 since the needed construction involved simply connecting two points. In this problem, the data were all consistent with the Perdix model. Successful solutions all appeared to involve pattern-based constructions at the level of general plans, with specific components of the plans worked out subsequently.

Problem 5

This problem was a relatively difficult theorem involving inequalities: "If two sides of a triangle are not congruent, then the angle opposite the longer of the two sides is larger than the angle opposite the shorter of the two sides." The problem is solved by a construction shown in Figure 17. Given $\triangle ABC$ with $AB > AC$, prove that $\angle ACB > \angle ABC$.

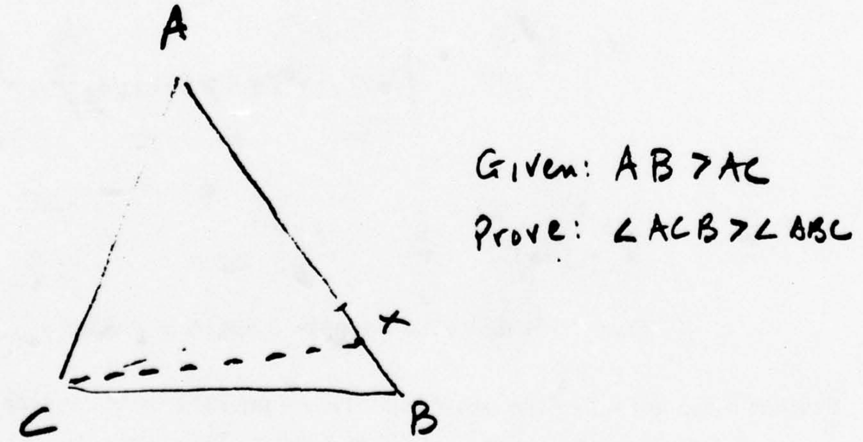


Figure 17. Diagram for solution of Problem 5.

First, find point X on AB such that AX = AC. (This point exists because AB > AC.) Construct CX. This forms an isosceles triangle in which $\angle ACX = \angle AXC$. Further, $\angle ACB > \angle ACX$ because $\angle ACB$ contains $\angle ACX$ as a part, and $\angle AXC > \angle ABC$ because $\angle AXC$ is the exterior angle of a triangle and $\angle ABC$ is one of the interior angles opposite $\angle AXC$. Therefore, we have $\angle ACB > \angle ACX = \angle AXC > \angle ABC$.

Subject 2. The protocol given by Subject 2 is in Table 12 and Figure 18. The subject's introductory remarks indicate that this subject remembered having proven this theorem, but did not remember the proof in detail. At *1, the subject added a line in the triangle and specified that it formed an angle $\angle X$ equal in measure to $\angle A$. This construction is invalid; in fact, any line drawn interior to this triangle from C will form an angle greater than $\angle A$. However, under the subject's hypothesis, a theorem does follow, which is drawn out at *2, *3, and *5 in the protocol. The subject failed to notice that the inequality proven is the opposite from the one in the goal of the problem. The subject and I continued for a few minutes talking about the construction, but did not make significant progress toward a correct solution.

Table 12

Protocol by Subject 2 on Problem 5

S:	Ah, let's see if I can remember this. (Pause.)
E:	What are you thinking about?
S:	Oh, I'm just trying to remember how to go about that. (Pause.) Okay. Now I draw a diagram. I probably won't remember. See, it's like . . . the way our teacher does it; you're given those theorems, and you have to prove them in your homework . . .
E:	Mmm-hmm.
S:	And then once you've proved them, you use them for the rest of your time, so you don't remember how to prove them.
E:	Okay.

S: However . . . okay. Let's see, AC is greater than . . . is that what you want? You want to prove that . . .

E: The angle opposite . . .

S: All right.

E: . . . the longer side . . .

S: Okay.

E: . . . larger than the angle opposite the shorter side.

S: Prove that angle B is greater than angle A. All right . . . hmm. (Pause.) Gee, I think that's . . . that's wierd.

E: What are you thinking?

S: Well, it doesn't . . . I don't think that works.

E: But what were you thinking about trying . . . ?

*1 S: Oh, I was thinking about trying . . . drawing an auxiliary line that was equal to this angle. So that this angle would be equal to this angle.

E: Oh, okay.

S: And then . . . let's see. And then saying that . . . let's say you have angle X here?

E: Mmm-hmm.

*2 S: Okay. X . . . the measure of angle X . . . is equal to the measure of the angle B plus measure of angle C. Because the exterior angle's equal to the . . . addition of the two interior remote.

E: Mmm-hmm.

*3 S: Okay, then saying X is equal to A . . .

E: Mmm-hmm.

S: That's given, or that was one of your conditions . . .

E: That's how you drew the line, right?

*4 S: And then you say the measure of angle X is greater than measure of angle B . . . or you could divide the measures.

E: Mmm-hmm.

S: Because the exterior, X equals B plus C . . .

E: Mmm-hmm.

S: And C is greater than zero, then A is greater than B . . .

E: Mmm-hmm. Good.

*5 S: And then since X equals A, you could just substitute in, and say that A is greater than B, but I don't think that's how we proved it the last time.

E: Okay.

S: Seems to work there.

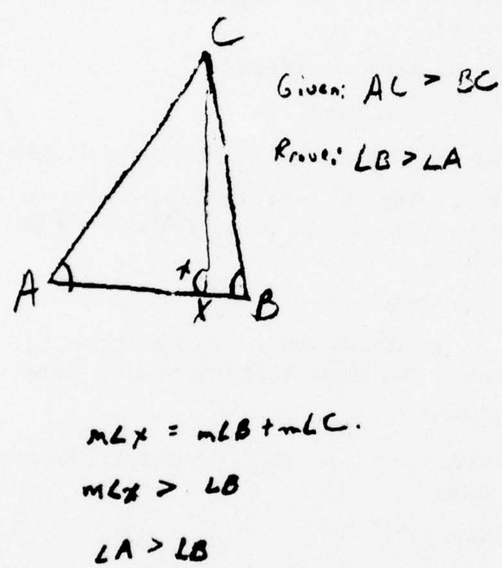


Figure 18. Drawing and writing by Subject 2 on Problem 5.

<u>Subject 3</u>. Subject 3's protocol is in Table 13. At *1, *2, and *3, the subject mentioned the triangle inequality. At *4, the idea of an exterior angle was mentioned. (The subject's diagram was drawn so that the angles to be proven unequal were $\angle A$ and $\angle B$.) At *5, the subject indicated that the theorem is intuitively reasonable--the remark seems to suggest a sense of the constraint dependency in this situation.

Finally, at *6, the subject searched for a relevant construction and did not find one.

Table 13

Protocol by Subject 3 on Problem 5

	S:	Okay. (Pause.)
	E:	What are you looking for?
	S:	I'm not sure yet. Angle and line relationships, I suppose. (Pause.)
	E:	Do you have any ideas?
	S:	Not yet. (Pause.)
	E:	Have you thought of anything you decided wouldn't work?
*1	S:	I can't find any relationships. I tried . . . the one about where you add the two sides and it's got to be . . . greater than . . .
	E:	Mmm-hmm.
*2	S:	. . . the third sides. And subtracting, it's got to be less than. You know what I'm talking about, don't you?
	E:	Mmm-hmm.
*3	S:	Okay. And . . . that, obviously, has nothing to do with the angles.
	E:	Okay.
	S:	So that doesn't make any difference.
	E:	Is there anything else you tried?
*4	S:	Well . . . let's see. The exterior angle of C equals angle A plus angle B.
	E:	Okay.
*5	S:	That's the only thing I've thought of. (Pause.) It has to be, but I can't explain why geometrically. Just because it has to . . . the two sides have to encompass the longer, you know, the greater distance.
	E:	Hmm.
	S:	Like that, instead of . . . that.
	E:	I see, mmm-hmm.

S: But . . . I can't say that, you know, that's certainly not . . . mathematical. That's not a formal proof of that.

E: Yeah, it has a problem with it, too, because the other angle that's involved can be . . . can affect the thing. If it was like an arc of a circle or something, then it would be a little stronger. (Pause.) Can you think of anything that you'd add to the diagram that might be of some help?

*6 S: Let's see. (Pause.) I could . . . what kind of auxiliary lines could I draw? I could draw a median. (Pause.) I could draw an altitude. (Pause.) Neither one of them seems to be terribly helpful. I could draw a line segment between the two midpoints. (Pause.)

E: Okay.

Subject 4. Subject 4's attempt to prove the theorem included drawing a triangle labeled in the same way as Figure 12, specifying that BC > AC. The subject drew a line from C to the base of the triangle and specified the construction as the bisector of $\angle ACB$. (Recall that Subject's 4's solution to the problem of proving that base angles of an isosceles triangle are congruent used the bisector of the apex angle.) Subject 4 also partially recalled a theorem about two triangles; if two sides of one triangle are congruent to two sides of another and the third sides are unequal, the triangle with the longer third side has the larger angle between the congruent sides. The subject actually misremembered the conslusion of the theorem as involving congruence and did not pursue that line.

Subject 5. Subject 5's protocol is in Table 14. At *1 the subject considered the fact that angles of a triangle sum to 180°, at *2 the triangle inequality was considered, and at *3 the subject considered and rejected the idea of drawing a median. Subject 6 also considered the triangle inequality, after remarking about "looking at it now to see if there's any lines I have to draw, any extra lines (pause) but I don't see any ones that I could draw that would really help me."

Table 14

Protocol by Subject 5 on Problem 5

E: What sort of thing are you looking for now? (Pause.)

S: Well, one thing I thought of, but it's not going to help me, is supplementary.

E: Ah.

*1 S: Well . . . not supplementary, but . . . I know that these three equal a hundred and eighty. And . . . one eighty minus these two equals

S (cont.): this, and one eighty minus . . . one eighty minus A and C equals B, and one eighty minus B and C equals A, right? But I don't see how that'll help me.

E: Okay. (Pause.)

*2 S: I know that . . . I know a theorem says that . . . AB plus AC is greater than CD.

E: Oh, right.

S: But I don't know how that would help me either.

E: Okay. (Pause.)

*3 S: An idea that I just thought of was to draw a median. I have to figure out where to draw a median. And then, by doing that, dividing up the triangle into congruent triangles. Well . . .

E: Can't think of anything to do?

S: No.

E: Okay.

Discussion. The attempts at solution by Subjects 3, 5, and 6 can be interpreted as failures to find a plausible plan. They retrieved propositions involving inequalities, but found no way to apply them to the situation. They also considered constructions that they knew could be made, but saw no way to develop a solution with them. These attempts are consistent, then, with the idea in Perdix that constructions are usually motivated by a solution plan for which prerequisites are missing.

Subject 2 apparently remembered the plan of proving an inequality using an exterior angle and also remembered a prerequisite for this situation involving the angle being part of an isosceles triangle. The construction made by Subject 2 had problems, but it did satisfy some relevant constraints. A reasonable interpretation is that Subject 2 remembered the plan of using an isosceles triangle consistent with Perdix's general process, but did not have adequate procedures for executing the plan.

Construction of the angle bisector by Subject 4 represents an interesting use of analogy, a process that Perdix cannot perform. This subject tried to proceed with this problem in the same general way that succeeds in proving congruence of angles when the sides are equal.

Problem 6

Problem 6 was, "Three sides of a quadrilateral have lengths of nine, four, and three. Between what values does the length of the fourth side lie?"

One protocol indicated that bounds on the length of a side of a quadrilateral had been considered in class in relation to the triangle inequality. This led two subjects to treat the problem in a way involving constructions. Two other subjects approached the problem with a direct spatial strategy, and one subject failed to make any substantial progress on the problem.

Subject 2. Subject 2's protocol on this problem is in Table 15 and Figure 19. At *1, the subject put a lower bound of zero on the fourth side. At *2, the subject made a construction by drawing the vertical line in the diagram. Bounds on the length of the constructed line were derived from the triangle inequality. At *3, the subject considered the triangle with sides of 9 and X. There was some

preliminary computation using the lower bound on X to obtain the lower bound on the fourth side and the upper bound on X to determine the upper bound on the fourth side. A qualification was considered; it is not clear what the subject meant by "this thing can hinge," but the conclusion is that the lower bound might be zero rather than eight. At *4, the subject realizes that another construction could be used and drew the horizontal line in Figure 19. This confirmed the subject's conclusion that the upper bound is 19.

Table 15

Protocol by Subject 2 on Problem 6

	E:	Three sides of a quadrilateral have lengths nine, four, and three. Between what values does the length of the fourth side lie?
	S:	Nine, four, and three?
	E:	Mmm-hmm.
*1	S:	Well, it's greater than zero, I know that.
	E:	Okay.
*2	S:	Let's see . . . nine, four, and three? All right. X here, all right. Oops. Four and three, okay, now I know that's . . . (pause) . . . one is . . . it's greater than one and less than seven.
	E:	Mmm-hmm.
*3	S:	And so if it's greater than one . . . then it has to be, if it's one, and this one has to be . . . eight. It has to be greater than eight, and it has to be less than sixteen. Now, but it can't be any number, I think, because this thing . . . can hinge. So, I think she had us replace it with zero. And it's less than sixteen.
	E:	Okay.
	S:	What I'm curious about right now . . . I always get off on these tangents when I do problems with you.
	E:	That's okay.
*4	S:	Whether it works the same with this. Because if it works the same with this, it would be five is less than X is less

than thirteen, so we have this five is less than . . . X is less than thirteen. And this one already . . . yeah, it works, sixteen.

E: Okay.

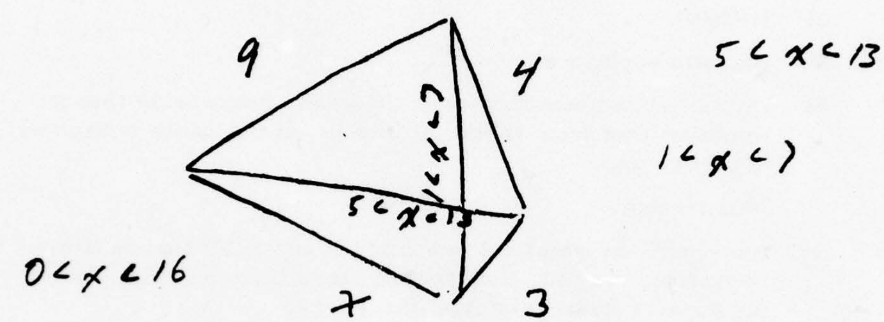


Figure 19. Drawing and writing by Subject 2 on Problem 6.

Subject 3. Subject 3's protocol on Problem 6 is in Table 16 and Figure 20. The construction was made at *1, bounds were put on the length of the auxiliary line at *2, and bounds on the fourth side were inferred at *3 (with a slight arithmetic error). The subject was relatively uncertain about the use of the bounds on the auxiliary line in the explanations given at *4 and *5.

Table 16

Protocol by Subject 3 on Problem 6

---

*1 S: Okay. I'll cut it into triangles.

E: Okay. (Pause.)

*2 S: That has to be . . . this one has to be . . . five less, and that's less than thirteen . . .

E: Mmm-hmm.

*3 S: This one has to be . . . (pause) . . . oh, that's strange. I guess it should be two . . . less than X . . . less than eighteen. (Pause.)

E: Mmm-hmm.

S: Is that . . .

E: Can you explain why?

*4 S: I'm not sure about this one. It seems reasonable that it would be five from three, if this is for the same reason as the other one.

E: Mmm-hmm.

*5 S: And then, the smallest you can get out of all that is three from five. Is two, less than X, less than eighteen, which is, oops, fifteen, sixteen, I'm sorry.

E: I see. Okay.

*6 S: I'm not sure, I' really not sure I did that right, but, well, I don't see why not, because that can be nine, and that can be four, and then, if that's three, there's no reason why that can't . . . you know, why it couldn't turn in like that, and like that. Well . . . it doesn't work out that well. But do you see what I mean?

E: Yeah, I think so. I'll say it so I'll be able to remember what the diagram was. You have a straight line for nine, and then a four, and then a three, and then, what is that sort of diagonal piece there?

S: I suppose . . .

E: In the middle of the kite?

*7 S: That's accidental. I suppose there's no reason why it couldn't be . . . oh, well, I never thought of it that way. It could be . . . concave. Quadrilateral. I never thought of it that way. I don't know what would happen to you then.

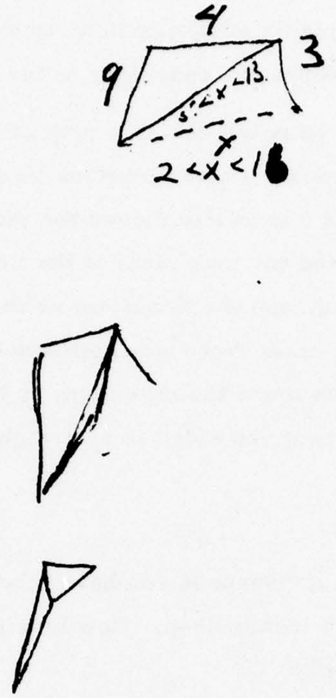


Figure 20. Drawings by Subject 3 on Problem 6.

Subject 3 engaged in some interesting spatial processing, mentioned at *6. The subject realized that the convexity of the diagram was not specified in the problem and explored the possibility of a concave quadrilateral. The subject apparently lacked a way of investigating that possibility and was left uncertain about the problem.

<u>Discussion.</u> Protocols from Problem 6 supported the general idea about constructions implemented in Perdix. These protocols are particularly convincing examples of the role of a pattern in generating constructions. The plan of finding bounds by using the triangle inequality requires that the unknown side be the side of a triangle, and the construction provides the triangle. Both subjects were quite uncertain

of how to execute the plan once they had the prerequisites, and Subject 3 explored the plan's adequacy in an interesting way. This is consistent with the top-down character of the planning mechanism.

It also should be noted that two subjects solved Problem 6 without using constructions. These solutions were based on semantic processing. Subject 5 mentally moved the sides of the quadrilateral that were given to find the maximum of the fourth side as the sum of the three given sides, and the minimum as the difference between the longest side and the other two sides combined. Subject 6 assigned a minimum of zero and found the maximum by the spatial manipulation of extending the three given sides in a straight line.

Problem 7

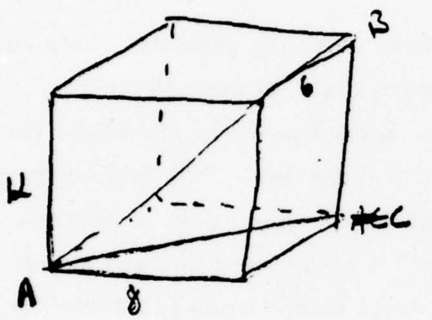
Problem 7 was, "Suppose you have a box that is 12 inches long, 8 inches wide, and 6 inches deep. How long is the longest stick you could fit into the box?"

This problem had not been worked by the class. The Pythagorean theorem was the current topic. Three subjects, Subjects 2, 3, and 5, found solutions quite directly. Each of them drew a clear diagram.

Subject 3. Subject 3's diagram is in Figure 21. The protocols are not very informative about the processes used by these subjects. A reasonable conjecture is that the longest dimension was found by some process of spatial reasoning, and its length became the goal of the problem. The right-triangle pattern was a salient plan for finding the length of a line. This requires that the target line is a side of a right triangle. To complete a triangle, connect an end of the target line with the end of another line that intersects with the target line.

Subject 3's retrospection was consistent with this hypothesis. When I asked, "Now, tell me how you decided to get that bottom line there," Subject 3 said, "AC? There wasn't any way I could just sort

of find it, just by going like that, without . . . with just the information given, because I only had . . . I didn't have . . . well, I had, I guess, one side to go on. I had its height. . . . But you've got to have both to use the Pythagorean theorem. . . . So, that means I had to have the other side."



$$64 + 36 = AC$$

$$\sqrt{100}$$

$$100 + 144 = AB^2$$

$$244 = AB^2$$

$$\sqrt{244} = AB^2$$

Figure 21. Drawing and writing by Subject 3 on Problem 7.

Subject 2. Subject 2's retrospection also fit with this hypothesis: "First thing was where would the biggest peice be. . . . Where would you stick in the stick? Diagonal from top to bottom. . . . And then I decided how . . . you had to find two sides that would have the hypotenuse being the whatever. . . . And I just took the bottom one, and so . . . using six and eight, I found the bottom one, and I found my two legs."

Subject 5. Similary, Subject 5 said, "I decided that I needed to find this first, which would be the length that I was after. . . . Then I looked for a way to find that out, and then I saw the triangle. . . . I saw the triangle here because I had to get a flat triangle, because . . . I just looked for ways . . . well, when I found this, I decided that I had to prove this, and to find out what that was I had to use this triangle right here."

Subject 4. Subject 4 was unable to solve the problem. This subject did not generate the goal of finding a diagonal through the box. The subject suggested the dimensions of the box (6, 8, and 12) as the length of the longest stick that would fit in the box. With further questioning, the subject asked, "Does this have something to do with the Pythagorean theorem, or something?" I encouraged this and asked the subject whether a stick longer than 12 inches could be fit into the box. The subject drew a rectangle with sides 12 and 8, constructed the diagonal, and mentioned the method of finding the length of the diagonal. The problem was eventually solved with considerable leading by me. A week later, the subject mentioned that the problem had been done in class on the day following the day of our previous interview. The subject did not remember the solution of the problem. This time, a three-dimensional diagram was drawn, but the diagonal through the interior was not generated.

Subject 6. Subject 6 did not solve Problem 7 but provided a very interesting protocol, which is in Table 17 and Figure 22. The subject began by drawing the diagram in the upper left corner of Figure 22. At *1, the subject apparently identified the goal of the problem correctly. However, Subject 6 did not work backward in the problem. At *2, the subject considered one of the rectangular sides and drew the rectangle in the upper right section of Figure 22. At *3, the subject apparently was considering the Pythagorean theorem as a way of generating the length of the diagonal. At *4, the second diagram on

the left was drawn, and at *5, its diagonal was inferred to be 10. At *6, the subject was apparently aware that the components of the problem had not been satisfactorily integrated. Perhaps this led to considering other components, and at *7, the subject computed the length of the diagonal of the 12 x 6 rectangle. At *8, there was an attempt to combine the results. The subject suggested that the diagonals of the two rectangles might be added to find the needed answer; see the line of writing at the bottom of Figure 22. At *9, the subject expressed uncertainty about the solution. There was not time to bring the situation into closure before we had to stop the interview.

Table 17

Protocol by Subject 6 on Problem 7

*1	S:	Now. Obviously the longest stick is going to be a diagonal from that corner to that corner.
	E:	Okay.
	S:	Or from that corner to that corner.
	E:	Okay.
*2	S:	All right, so . . . there are certain things that we can say about this . . . all right, so let me take one of the sides. It's obviously going to be . . . first of all . . .
	E:	What did you have in mind . . .
	S:	Twelve by six.
	E:	. . . when you said . . .
*3	S:	Well, I know one thing about . . . Pythagorean triples, I think . . . yeah, that's what you call it. That when you go like that . . .
	E:	Yeah.
*4	S:	All right, and I know . . . that's eight.
	E:	When you say like that, you mean the diagonal?
	S:	Right. You draw the diagonal, which would be the stick . . .
	E:	Mmm-hmm.

*5 S: Eight, six, so the diagonal is going to be ten.

E: Ah. Okay.

S: Because of Pythagorean triples.

E: Mmm-hmm.

S: I could figure . . . you could figure it out, you know, it works out. But now, I've got to take in the three-D idea . . .

E: Right.

*6 S: And . . . I've got to do on this . . . eight . . . eight by six, and draw the diagonal, and that is going to . . . mmm-mmm. All right, I figure out the distance. The first diagonal, I figured out the distance . . . from . . . okay. Regroup here. I've got . . . let's make sure I've got my things right here. That is twelve.

E: Okay.

S: Now . . . all right, so . . . let's see . . . so I've got twelve and six on the two sides . . .

E: Yeah.

*7 S: So . . . Pythagorean theorem is twelve squared is a hundred and forty-four . . . plus thirty-six is equal to . . . whatever . . . whatever this C squared.

E: Right.

S: So . . . I could say a hundred and forty-four and thirty-six is going to be . . . let's see, that would be zero, five, and three . . . is going to be eight hundred. Is that right? Yeah, okay, a hundred and eighty . . . is equal to C squared. So C is equal to the square root of a hundred and eighty, and that looks like . . . I'm not sure if that's a perfect square or not. (Pause.) Well, do you want me to simplify that now, or just leave it like that?

E: Whichever way you want to go ahead.

S: All right. Well, all right, I'll see what happens then. I have one with six and eight . . . and if I draw that one, this is also going to be eight, and this is going to be six, so thirty-six plus sixty-four is equal to C squared. So, it's going to be equal to ten here. Because that's a hundred, C squared is equal to a hundred.

E: Mmm-hmm.

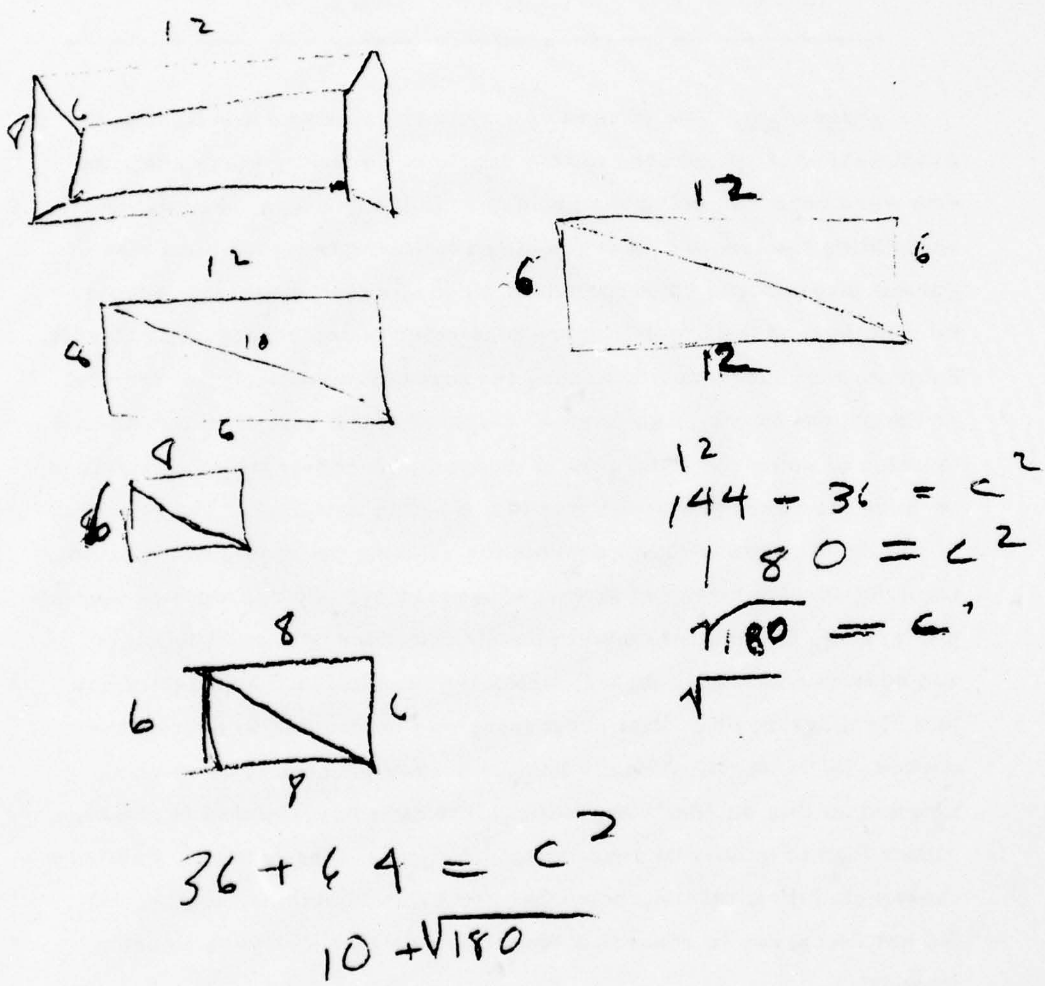


Figure 22. Drawing and writing by Subject 6 on Problem 7.

*8 S: Okay. So, I would say that the way to find the three-D would be to add that one . . . add that one, add the length of . . . this, plus the length of . . . plus the eighty. And the square root of one eighty simplified . . . let's see . . .

E: No, that's okay. That's close enough. Okay.

*9 S: You know, I may be, I probably . . . I don't know, I think that's the way you do it, but I'm not sure.

Discussion. The pattern of a right triangle was a salient part of all subjects' approaches to this problem, including those subjects who were unable to solve the problem. In those cases, the goal of calculating the length of a segment led to attempts to use the Pythagorean theorem and construction of diagonals of rectangles. Successful solutions of the problem were consistent in one respect with Perdix. Subjects who succeeded in solving the problem apparently set the goal of finding the length of an internal diagonal of the box, then considered the plan of using the Pythagorean theorem. A construction was needed because the prerequisite for this plan was not satisfied. The process of finding a construction to provide the missing prerequisite apparently involved some interesting semantic processing. By appropriate spatial processing, the subjects could identify two sides of a right triangle and construct the missing leg, which led to solution. Apparently Subject 6's diagram or spatial processing was inadequate to specify the needed construction, although Subject 6 often proceeded by working forward so this subject's approach might have been related to strategy rather than to quality of representation. In working forward, Subject 6 constructed diagonals in rectangles and calculated their lengths, but did not find a way to connect diagonals of the faces with the interior diagonal.

Problem 8

I gave the problem of proving that an angle inscribed in a circle has measure equal to one-half the measure of the arc intercepted by the angle. This theorem had not been covered in the course when I gave it initially.

A simpler problem, shown in Figure 23, was given to most of the subjects. Two subjects who did not prove the theorem initially were given this problem and then tried the theorem a second time. With one subject, I gave the problem of Figure 23 initially and then presented the theorem.

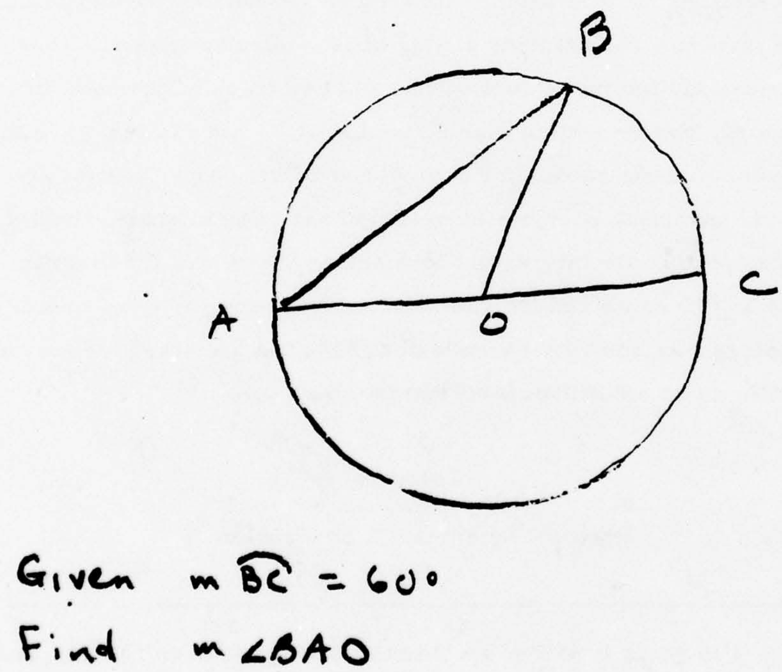


Figure 23. Auxiliary problem given in relation to Problem 8.

One week later, I presented the theorem a second time. The students had studied this theorem in class during the interim.

Subject 2. On Problem 8, Subject 2 found a proof directly for the case where sides of the angle are on opposite sides of the diameter of the circle. The protocol for Subject 2 is in Table 18 and Figure 24. At *1 and *2, the subject conjectured correctly about the concept of an inscribed angle. At *3, I clarified what was meant by the intercepted arc. At this point, the subject had drawn the circle at the top of Figure 24 with lines AC and BC. At *4, the subject marked the central angle for arc $\widehat{AB}$ and indicated a goal of relating it to the inscribed angle. At *5 and *6, the subject identified AO and BO as radii, and at *6, the subject constructed the diameter of the circle and identified CO as a radius. At *8, the subject apparently had noticed the critical relation involving the exterior angles of isosceles triangles. There were apparently too many components to keep track of them all in working memory, so the subject labeled angles at *9 and *10 and gave their relationship and the reason for it at *11 and *12. After a brief diversion at *13 involving possible bisection of the central angle, the subject made the appropriate inference about angles y and k in the diagram at *14, and at *15 combined the two inferences to complete the proof. Terminology was somewhat confused at *15, but got straightened out at *16 with some additional labels in the diagram.

Table 18

Protocol by Subject 2 on Problem 8

E: I'm going to tell you a theorem that you haven't gotten to yet, and I just want to see whether you have any idea about how you might try to prove it. Do you know what it means to have an angle inscribed in a circle?

S: Inscribed in a circle?

E: I don't think you can . . .

S: I could guess. No, I don't.

E: Just tell me what your guess is.

*1 S: It's an angle that . . . has the vertex being one point on the edge of the circle.

E: Mmm-hmm.

*2 S: And . . . its . . . well, an angle really doesn't have any endpoint . . . its endpoints are . . . the two legs are chords in the circle.

E: Yeah. That's right. That's exactly right.

S: Okay.

E: Now, here's the theorem. The arc intercepted by an inscribed angle has twice the measure of the inscribed angle. (Pause.)

S: What's intercept?

E: That means the arc between the two . . . between the two points where the sides of the angle intersect the circle.

S: All right. The arc intercepted by the angle.

E: Yeah.

S: This right here.

E: No, it's . . .

S: This one right here?

*3 E: Yeah. The one between the sides.

S: Is twice . . .

E: Mmm-hmm.

S: . . . the measure of this?

E: Yeah.

S: Oh, I see. Okay.

E: Now, do you have any idea how you might prove that?

*4 S: Well, the way I look for the proof is I'd say . . . I'd first see if I can find out if I can relate . . . this angle to this angle.

E: Mmm-hmm.

S: And prove that this angle is twice the measure of this angle.

E: Okay, now the angle you're going to relate to it is the central angle for that arc.

S: In fact, I think I can tell you.

E: Sure.

*5 S: This is a radii, right?

E: Those are radii.

*6 S: And this is a radii.

E: That's right.

S: Radius. These three are radii.

E: Good.

*7 S: Okay. This is equivalent to this. Therefore, it's an isosceles triangle. This is congruent to this.

E: Okay.

*8 S: Right? (Pause.) Now wait. What I was thinking of . . . you know, I was thinking of taking . . . this would be . . . this would be equal to twice . . .

E: Okay, now . . . so that I can untangle this at the end . . .

S: Yeah.

E: What do you mean by "this" now? Why don't you put a little number or something?

*9 S: Okay. We'll call this angle X.

E: Okay. Angle X.

* 10 S: We'll call this J, all right?

E: Okay.

*11 S: X is equal to two J.

E: Okay, now why is that?

*12 S: Because it's a . . . because an exterior angle is equal to the sum of the remote angles. And since these two are angled by an isosceles.

E: That's right. Okay.

S: Okay. Angle Y . . .

E: Okay. (Pause.)

*13 S: Is equal . . . these two would also be equal. You know, this . . . (pause) . . . what I'm not sure of is whether this radius would bisect the angle.

E: It might, but it wouldn't have to.

S: It wouldn't have to.

E: It would depend on the degree of that angle.

S: That's right, that's right. Okay. (Pause.) Two J . . . Y would equal . . . let's call this K, okay?

E: Okay.

*14 S: Y would equal two K, is equal to two K.

E: Mmm-hmm.

*15 S: Okay. (Pause.) Therefore, X plus Y would equal two K plus Y. Since X plus Y . . . let's give these letters, okay? We'll call the center O, call this A, B, and C.

E: Mmm-hmm. Okay.

*16 S: All right. AOB . . . angle AOB . . . is equal to X plus Y, so you just substitute in here, and angle ACB is equal to K plus J. K plus J, I'm sorry.

E: Mmm-hmm.

S: So, angle . . . the measure of angle ACB . . . therefore . . . therefore angle AOB is equal to twice . . .

E: Mmm-hmm.

S: . . . the measure of angle . . . ACB.

E: Mmm-hmm.

S: And therefore the arc is twice as big.

E: Okay.

S: Okay.

*17 E: That works any time you've got both sides of the angle on . . . when the two sides of the angle are on opposite sides of the diameter.

S: That's right.

*18 E: In fact, it would work fine if one of the sides was the diameter.

S: Yeah. What would happen . . .

*19 E: What would happen . . . you know the other question.

S: All right. (Pause.) All right, we have O here.

E: Right.

S: Now, okay. You're saying this is equal to twice?

E: That's also twice its . . .

S: Okay. (Pause.) How to go about setting up . . . the theorem. (Pause.) If I draw that across . . . I'll tackle it the same way.

E: Okay.

*20 S: See if I can get any results with that. All right. They both have . . . let's see. Both these angles have the same chord . . . two endpoints of the arc. I'm trying to prove that AOB is twice . . . angle AOB is twice the length . . . or, twice as great as angle ACB.

E: Mmm-hmm.

S: All right, now. (Pause.) What I'm looking for now is I'm trying to see if I can draw any auxiliaries here that will relate . . .

E: Mmm-hmm.

S: . . . to triangles . . .

E: Mmm-hmm.

S: To get . . . (pause) . . . gosh. (Pause.) Maybe I should erase this.

E: Maybe it would be easier to just start over.

S: Yeah. (Pause.) Worse than the first. (Pause.) All right, if I draw from O to B . . .

E: Mmm-hmm.

S: And O to C . . . I'll have a side for each of these triangles.

E: Okay.

S: All right? (Pause.) I still don't get anywhere. (Pause.) Gee, I don't know.

At *17, *18, and *19, I asked the subject to consider the case where both sides of the angle are on the same side of the diameter of the circle. The subject drew the second diagram in Figure 24. At *20, the subject considered the central angle, $\angle$ACB, trying to relate it to the inscribed angle. Further work led to constructing the third diagram in Figure 24, with labels as shown. I helped the subject solve the problem with the following hint: "How is angle ACD related

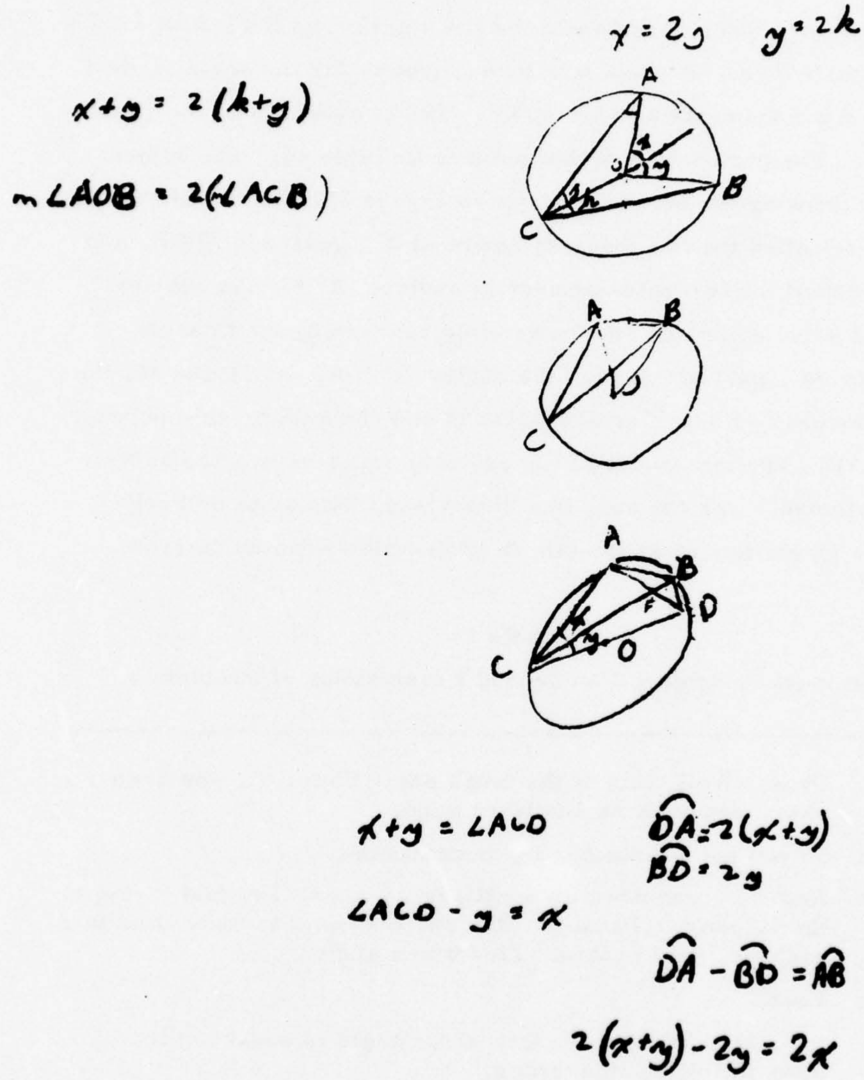


Figure 24. Drawing and writing by Subject 2 on Problem 8.

to angle X and angle BCD?" This led to the solution fairly directly, although with considerable thought along the way.

In the second presentation a week later, the subject remembered that there were three cases and drew the diagram on the left of Figure 25 to identify them. A proof was found directly for the angle formed by XO and the diameter and for $\angle OXP$. Then I asked about the remaining case. The protocol from that point is in Table 19. The subject began by drawing the second diagram in Figure 25. At *1 and *2, the subject identified the two relevant angles at X, $\angle ACD$ and $\angle DCB$, and also identified the two relevant central angles. At *3, the subject indicated some confusion and uncertainty concerning the triangle formed by AC, part of AD, and the radius from A. At *4 and *5, the subject focused on arc $\overset{\frown}{BD}$ and $\angle BCD$ and saw the relationship between BD and AD. The remainder of the protocol suggests that the subtractive relationship was not seen in a direct way, but had to be derived using the formalism of arithmetic in conjunction with the diagram.

Table 19

Protocol by Subject 2 on Second Presentation of Problem 8

	S:	Okay. Well, this is the tough one. Uhm . . . you draw . . . okay, you draw an auxiliary here.
	E:	So you put a diameter in, mmm-hmm.
*1	S:	And . . . you draw an auxiliary . . . no. I'm just trying to think, okay. (Pause.) This angle's equal to this. And this angle is equal to this. This whole angle . . .
	E:	Yeah?
*2	S:	. . . is equal to . . . this whole angle is equal to this. I think I'm doing this wrong.
	E:	The whole angle . . . I need to slow up with you. Those two central angles . . .
	S:	Yeah. Is equal to this plus this. (Pause.)
	E:	Okay.

S: Oh, since these are congruent . . . oh. (Pause.) There's something wrong. (Pause.) For me the second time is no better than the first.

E: That's an interesting problem. What makes you think you've gotten off the track?

*3 S: Because I'm working with this little triangle right here, and I'm really not sure I wanted to.

E: That little triangle?

S: Right here. This little triangle that's sitting here.

E: Okay.

S: And it's an area that I'm not sure I want to fool around with.

E: I see. (Pause.)

S: This right here . . . is equal to this arc. This angle right there . . .

E: That's right.

S: . . . is equal to this arc. And these two are congruent . . . so why shouldn't . . . if you . . . (pause) . . . these two are congruent.

E: Yeah.

S: *Since this* . . .

E: No what's congruent here?

*4 S: Oh, no, these aren't congruent. These aren't congruent. I see. Now . . . (pause). I've already proved that this arc . . . A to B . . . is equal to angle C plus angle AO, okay?

E: Okay.

*5 S: All right. Or angle CAO. So, okay . . . therefore AB is, yeah, it's twice the amount of angle C. Now, I want to subtract BD, and angle DC to B . . .

E: Right.

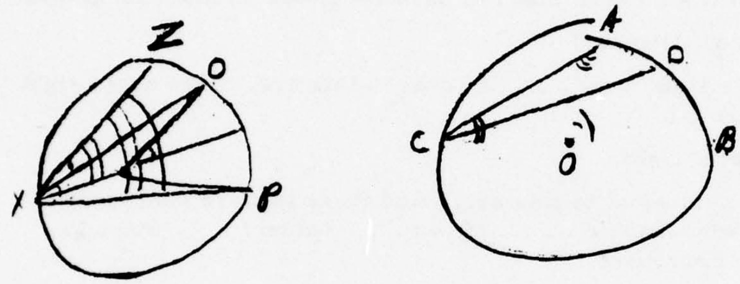
S: And end up with AD being twice as much as this. I attack it from the . . . yeah, wait a minute. Okay. (Pause.) Oh, okay, okay. Okay, angle ACB is . . . the measure of angle ACB is equal to two times the measure of arc AB, right? Now, if I . . . (pause) . . . if I subtract . . . (pause) . . . measure of angle ACB . . . equals the measure of angle DBC . . . which will equal . . . this one . . . AB . . . (pause). Is that right? (Pause.)

E: So you have two there, and then . . .

S: Mmm-hmm. No, because I just took this. Oh, that's not right, is it? Oh, angle . . .

E: Angle ACB, I think.

S: ACB . . . angle DCB is equal to two . . . okay, there you go. So, then you just use the addition or subtraction property of angles, and you take the two out using the distributive property, and then you've got it.



$$m\angle ACB = 2\,m\overset{\frown}{AB}$$

$$m\angle ACB - m\angle DCB = 2m\overset{\frown}{AB} - 2m\overset{\frown}{AD}$$

Figure 25. Drawing and writing by Subject 2 on the second presentation of Problem 8.

Subject 3. The protocol with Subject 3 began with an explanation of an inscribed angle and its intercepted arc. After thinking for awhile, the subject said, "The first thing I thought of was a triangle, like that," and drew an angle on opposite sides of the diameter of a

circle, connecting the ends of the angle by a chord. Then the subject went on, "But that doesn't really give me the measure of the arc. That's . . . I mean, completely different. So then, I started to think about central angles, and I haven't gotten anywhere with that yet. And I don't think it's really going to work, so . . . you know, what if your angle was like that, and there's no way either one of those is going to be a diameter, and so, neither one of those is going to have the central angle." The subject had drawn another angle with one of the lines, having both sides on the same side of the diameter of the circle.

I decided that the subject was unlikely to consider special cases without a suggestion, so I suggested the case where one side is a diameter. The subject proceeded directly to the proof for that case, including construction of the central angle, mention of the isosceles triangle formed by the two radii, and the exterior angle of the triangle.

Next, I suggested the case in which sides of the angle are on opposite sides of the diameter. After drawing an appropriate diagram, the subject said, "Oh, oh, hey. I know what I can do. I can use chords. Maybe. I mean, chords is just a whole new idea that just occurred to me. I hadn't thought about using chords. What do I know about chords? Not a whole lot. Another thing is that the angles, they don't have to . . . those chords don't have to be congruent. And most of the stuff we know about chords is about congruent chords. . . . Triangles don't do much good, that's for sure." The subject was apparently thinking about forming a triangle inscribed in the circle and may have been considering possibilities such as isosceles or equilateral triangles. Then the subject said, "Draw a diameter through there, and then do the same thing again, I suppose." This was the solution; however, the subject required a few steps of quantitative reasoning that were quite interesting. "I was just wondering if my idea was right. But, in this case it looks like these two angles are congruent. The two angles . . . They wouldn't have to be. But it probably doesn't matter,

because if they're not, you can say, you know, like, double one-third. And double two-thirds, and it'll still add up to be the same. . . . You have double whatever that fraction is of those two, and that equals that. And double whatever fraction that is, and that equals the other one. And then add the two fractions together and you have the whole, and then you add the two pieces together there and you have twice that. I mean, it's the same deal."

In considering the third case, the subject apparently was not making progress. I provided the following hint: "The second one we did was based on the first one. By seeing that you could get the angle that you wanted by adding two angles, each of which involved a diameter. You got the angle on this side that involves the diameter, and you got the angle on this side that involves the diameter. So the inscribed angle turns out to be the sum of two angles that have the diameter. Does that give you any ideas?" The subject asked to have it repeated, in which I said, "Each one of them involved an angle, you add together two inscribed angles that are on opposite sides of the diameter." Then the subject said, "Do you mean that I could find this one and subtract?", and worked out the proof.

Subject 3 solved the problem in Figure 23 directly and easily. Unfortunately, I did not obtain an interview from Subject 3 the following week.

Subject 5. When the theorem was presented to Subject 5 initially, little progress was made. I explained what an inscribed angle is. The subject drew two diagrams containing inscribed angles and, in both of them, completed triangles by drawing chords opposite the vertices of the inscribed angles. The subject expressed some confusion about the theorem: "If you have a circle, and then you have an inscribed triangle. Now this arc is the same as this angle. You're saying it's twice. That's why I'm confused." I do not understand the nature of the confusion. One possible explanation is that the class had studied

central angles, and the subject may have been confusing the inscribed angle with the central angle.

On the problem in Figure 23, Subject 5 initially solved for the wrong angle, saying, "Okay. BOA . . . okay, BC. This equals sixty," writing 60 in the central angle $\angle BOC$, then, "BOA, that's easy. That equals . . . a hundred and eighty," and wrote 120° in $\angle BOA$. When I pointed out that the problem asked for $\angle BAO$ the subject said, "BAO, I'm sorry. Okay. If I had the tangents, I could use . . . kind of, because . . . I know this is a hundred and twenty. A hundred and twenty degrees." Here the subject marked 120 on the chord $\overline{AB}$. "I'd better get that straight in my head. Oh, well, that'll make it easier. Because a hundred and twenty minus . . . This is isosceles . . . Okay, so it'd be . . . a hundred and twenty, this is one eighty, sixty. And, let me think . . . I have sixty degrees to go. So these are thirty." It is not clear where the insight about the isosceles triangle came from, although it seems likely that it had to do with the subject's consideration of tangents. A plausible sequence is that the subject may have considered the central angle, thought of tangents to the circle, noticed the radii perpendicular to the tangents, and remembered that they were congruent.

We returned to the theorem. The subject draw a diagram as I stated the theorem, then said, "So this is two J and that's J." and marked the arc and inscribed angle. The subject again completed the inscribed triangle. The subject said, "What I was thinking of doing is . . . radii . . . and . . . to bisect. But see, the thing is . . . the thing about doing that . . . is I could bisect the arc, like that. Or I can draw out to the sides." The subject drew radii to the three points on the circle and noted that they were all congruent, forming isosceles triangles. I discouraged the specification of the bisector of the angle. The subject then drew a diagram with inscribed angle in one semicircle, and I discouraged that.

The subject seemed not to be making progress, and I suggested trying the case in which a side of the angle is a diameter. Once again, the subject drew an inscribed triangle and marked the inscribed angle J and the arc and central angle as 2J. The subject also noted the angle formed by the sides of the inscribed and central angles was J. All the pieces seemed to be available, especially when the subject said, "Well, I can prove that . . . I was thinking of remote angles." However, the structure seemed not to form. Among the following remarks were, "That won't prove it. That'll just find out the measure," and "Maybe what I should do is I should try to get them in the same triangle." I eventually pointed out the relationship, which the subject accepted but still seemed to lack a clear grasp of the situation.

The next week when I brought up the theorem again, the subject had seen it in class and recalled the theorem but did not remember seeing the proof. The subject drew an inscribed angle with sides in the two semicircles of a circle, drew the central angle for it, drew a diameter and connected its end with the other two points on the circle, and mentioned that the angles inscribed in the semicircle were right angles. However, there seemed to be little progress toward a proof of the theorem.

I suggested that the subject work on the case with one side of the angle on the diameter. The subject drew a diagram and once again completed an inscribed triangle and a radius to form the relevant central angle. The subject said, "These two are supplementary and . . . that makes this equal to these two," referring to the central angle, its supplementary angle along the diameter, and the inscribed angle along with the other base angle of the isosceles triangle. I said, "What do you know about that exterior angle compared to the arc?" The subject said, "They're the same. Oh, so if I could prove these . . . oh, okay. Then these two are radii, so they are congruent . . . Okay, so I can figure that out . . . Okay, so that makes . . . this

one half . . . because these two angles are congruent, and this is congruent to these two. . . . So that makes two times . . ."

I suggested working on the second case, with the angle in two semicircles. The subject drew an angle with both sides in one semicircle. When I pointed this out, the subject drew an appropriate diagram, completing the inscribed triangle, and drew a radius of the circle through the chord opposite the inscribed angle. There was some discussion about how this bisector would not generally extend to the vertex of the inscribed angle.

I drew a diagram with the sides on opposite sides of the diameter and added the diameter. The subject drew the chord from the end of the diameter to one end of the inscribed angle and spelled out the proof that the arc is twice the inscribed angle formed by the diameter and one side of the initial inscribed angle. The subject then added the remaining radius to form the central angle on the other side of the diameter (<u>not</u> adding the chord, for the only time), showing the relation of that central angle to its corresponding inscribed angle, and then working out the algebra to show that the complete inscribed angle was one-half of the sum of the two arcs.

On the third case, a solution was worked out, but it involved considerable guidance, including my suggesting use of the diameter, suggesting that the diamter was the side of two angles, and shading the part of the central angle that was to be subtracted to obtain the central angle for the initial inscribed angle.

<u>Subject 6</u>. Subject 6 did not find a proof of the theorem initially. The subject drew a diagram and included a diameter and a chord from the two ends of the inscribed angle. The subject mentioned some properties of congruent arcs and mentioned central angles, but did not construct a central angle.

Subject 6 solved the problem in Figure 23. The subject noted that OB was a radius, inferred that $\angle BOC$ was 60°, and inferred that $\angle AOB$ was 120°. Subject 6 mentioned that a triangle formed by a radius and a chord was equilateral. After some thought, the subject inferred that the two outer angles in the triangle were equal. This led to the solution and a remark that the subject really had meant that the triangle was isosceles, rather than equilateral.

Returning to the theorem, the subject began with the angle sides on opposite sides of the diameter and was not making progress. I suggested making one side of the angle a diameter. The subject said, "Ah-hah. You make a central angle from A to O." The subject specialized the problem by making the central angle 90° (although it appeared to be about 80°). The inscribed angle was then found to be 45°, and the subject noted that this was inferred from the base angles of the isosceles triangle, formed by two radii.

When the theorem was presented again a week later, the subject recalled its being covered in class the previous day and drew a diagram for it. The subject remembered that there were several cases and that the first case involved a side of the angle as the diameter of the circle. The subject also recalled that the proof involved use of a triangle. The triangle that the subject constructed was formed by adding the chord of the circle to connect the ends of the angle. The subject noted that this was a right triangle, but was unable to proceed further.

I suggested that the triangle the subject had constructed was not the one needed to prove the theorem. This was sufficient for the subject to construct the triangle formed by the radius of the circle. The subject constructed a sketch of the proof of the theorem, using congruent angles based on the isosceles triangle (the subject mentioned the radius and chord of the circle), supplementary angles at the center of the circle, and the equality of the central angle with its intercepted

arc. The subject then recalled the other two cases and proved them by relations of addition and subtraction of angles and arcs.

Subject 4. I gave the problem in Figure 23 to Subject 4 without presenting the theorem first. The subject was unable to completely solve the problem independently. The measure of $\angle BOC$ was inferred, as was $\angle BOA$. Then the subject conjectured that the remaining angles of the triangle were congruent, but did not find a justification for this. I eventually directed the subject's attention to the fact that the sides of the triangles were radii, making it an isosceles triangle, which made the angles congruent.

I defined an inscribed angle and Subject 4 drew an appropriate diagram with the diameter between the sides of the angle, but the subject did not have definite ideas of how to proceed. The following week, Subject 4 was able to state the theorem, but was not able to make substantial progress toward proving it. For the case with a diameter as one side of the angle, I was able to prompt the construction of the relevant central angle by asking about an angle equal to the arc. The subject considered the triangle at some length, including the possibility that it might be a 45-45-90 triangle as well as that it might be isosceles. When I encouraged the idea that the triangle was isosceles, the subject worked out the argument that the inscribed angle was one-half the central angle, using the sum of angles in the triangle and the supplementary relation of the angles at the center of the circle. I suggested that the subject might be able to decide whether the triangle was isosceles, and the subject identified the sides as radii. It was not clear that the subject had a strong grasp of the proof; the subject expressed some uncertainty after getting all the components of the solution.

Discussion. The problem is evidently hard in several ways. The theorem must be analyzed into cases, which is a motivational step and one that Perdix cannot perform. Once the first case is

generated, the construction of the central angle must be included. Then the relation of this angle to the isosceles triangle and the inscribed angle as one of the base angles of the isosceles triangle must be seen. This complicated set of relations was seen independently by two of the subjects, but not the others. However, it was understood after study of it in class by all but one of the subjects. We can conjecture that the class study may have led to the students having a planning structure like those programmed in Perdix. Extension to the case involving addition seemed much easier than to the case involving subtraction. This suggests that the spatial operation of angle addition may be much more easily performed than angle subtraction, at least in the configuration involved here where addition involves a simple concatenation of both the inscribed and central angles, but the central angle for the inscribed angle in one semicircle is not contained in the inscribed angle, creating a more complicated spatial configuration.

Problem 9

The final problem involving constructions involved an extension of the well-known problem of finding the area of a parallelogram. The Wertheimer (1959) transformation had been presented in the class. I presented a related problem: finding the area of a trapezoid. The diagrams shown with this problem are in Figure 26.

I presented Panel (a) of Figure 26 to Subject 2 and asked the subject to recall the formula for the area and give a proof of the formula. ($A = \frac{h}{2}(b_1 + b_2)$ where h is the attitude and b_1 and b_2 are the bases.) With the other subjects, I began by asking for calculation of the area of a trapezoid with numerical values for the bases rather than for proof of the formula. If the subject used a formula for the area, I asked for justification. In presenting the problem, I omitted to specify the altitude

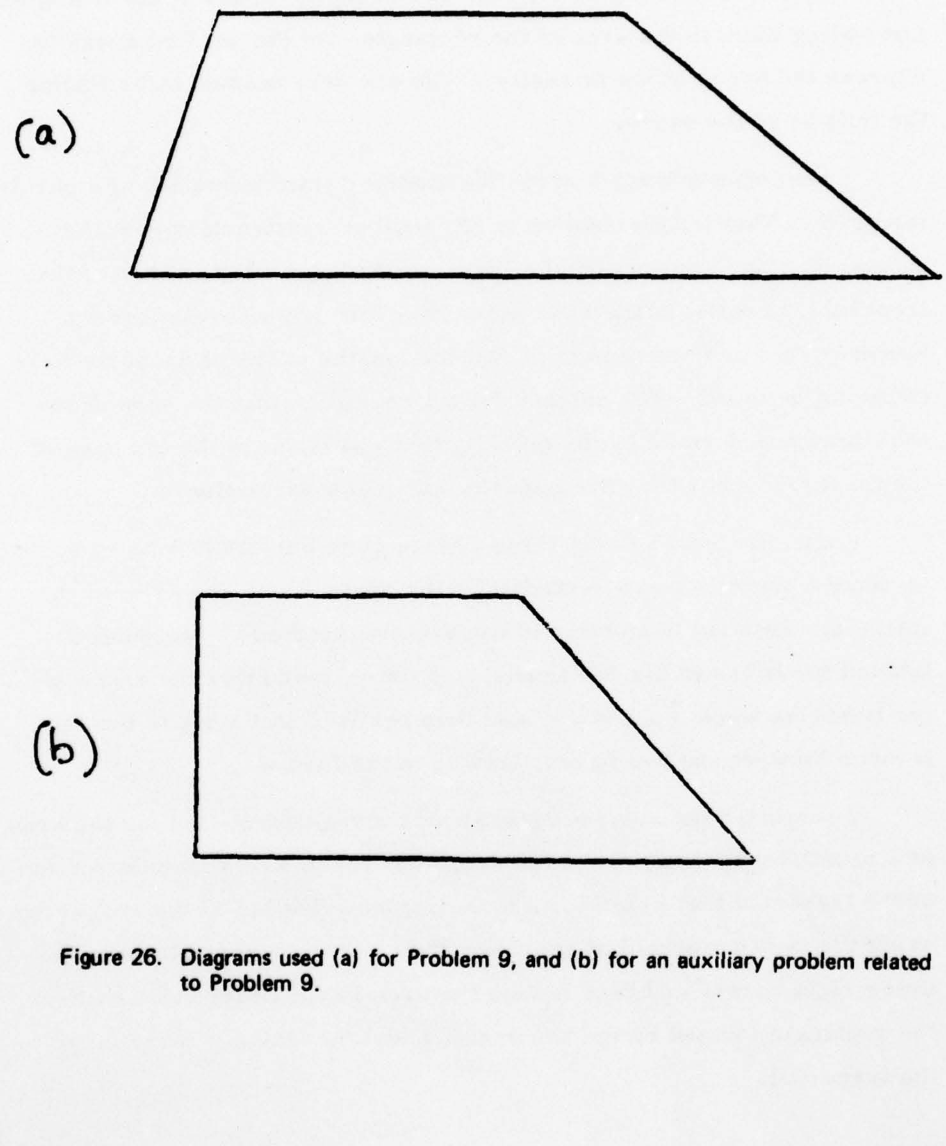


Figure 26. Diagrams used (a) for Problem 9, and (b) for an auxiliary problem related to Problem 9.

of the trapezoid. I thought that presenting the altitude and the bases might provide too strong a cue for recall of the formula for the area.

Subject 2. The subject retrieved the correct formula and, when asked for a proof, dropped perpendiculars from the two upper corners. The subject was working on the idea of finding the areas of the triangles and adding them to the area of the rectangle, but did not find a way to express the areas of the triangles. The problem seemed to be finding the lengths of the bases.

I reminded Subject 2 of the Wertheimer transformation of a parallelogram. This led the subject to try another construction with the trapezoid. The subject added a line from the upper left corner of the trapezoid, parallel to the right side, thus forming a parallelogram. However, the subject remarked that the lengths of the parts of the base could not be found. The subject did not recognize that the base of the parallelogram formed by the construction was equal to the top base of the trapezoid since they are opposite sides of a parallelogram.

I then presented Panel (b) of Figure 26 to Subject 2, who constructed a triangle and a rectangle in the obvious way and derived the formula. This led to solution of the original problem. The subject labeled the bases of the two triangles x and y, noted that the areas of the triangles were $\frac{h}{2}x$ and $\frac{h}{2}y$, and then realized that $x + y$ is the difference between the two bases, leading to the formula.

I reminded the subject again about the transformation for the area of a parallelogram and asked the subject to try to find a transformation of the trapezoid that would form a rectangle. This led to the suggestion using the right trapezoid of extending the top base to a point above the lower right corner and then finding the area as the differences between the rectangle formed by the construction and the triangle exterior to the trapezoid.

Finally, the subject recalled a transformation involving a horizontal partition through the center of the trapezoid, with a 180° rotation to form a parallelogram. The height of the parallelogram is $\frac{h}{2}$, and the base is the sum of the bases of the trapezoid, giving the formula.

Subject 3. Subject 3 calculated that area of the trapezoid using an impressive transformation of the figure. A construction was performed, involving two auxiliary lines that formed a rectangle and two triangles. Then the two triangles were considered as a single triangle, with base equal to the difference between the trapezoid's bases. This was done numerically, without use of a formula.

I presented a parallelogram and asked Subject 3 to recall the formula for its area. The subject recalled the formula and volunteered the justification involving translating a triangle.

Next I presented the right trapezoid in Panel (b) of Figure 26. I asked the subject to give a formula for its area. The subject labeled the lower base b_2, the upper base b_1, and the altitude h. The subject's first attempt at a formula was $A = h\,(b_2 - b_1) + 1/2h\,(b_2 - b_1)$, but the subject corrected this to $A = hb_1 + 1/2h\,(b_2 - b_1)$. The subject expressed interest in simplifying the formula, but did not see a way to do it. The subject and I worked through some algebra to arrive at $A = 1/2h\,(b_1 + b_2)$.

I returned to the original problem, Panel (a), and asked the subject to show that "the same thing would work." The subject labeled the bases and height in the same way as in the right trapezoid and labeled points A, B, C, and D along the lower base. Then the subject remarked that $b_2 - b_1$ would equal AB + CD, the length of the base of the triangle formed by combining the two triangles from the ends of the trapezoid. Then the formula $A = hb_1 + 1/2h\,(b_2 - b_1)$ was derived.

I presented the right trapezoid again. I reminded the subject of the Wertheimer transformation and asked whether the trapezoid could be transformed into a rectangle. The subject was quite skeptical. The first attempt involved drawing a line parallel to the right leg of the trapezoid, starting at the upper left corner, but that was not judged to make progress toward a rectangle. When I emphasized that what was wanted was a rectangle the same area as the trapezoid, the subject said that would be "a little fatter," and then constructed a line midway between the ends of the two bases. The transformation was described: "Chop off that little piece and add it to that." The subject also worked out the formula for the length of the rectangle, starting with $b_1 + 1/2\ (b_2 - b_1)$ and deriving $1/2\ (b_1 + b_2)$. When I asked about the area, the subject wrote the expression $h\ 1/2\ (b_1 + b_2)$.

I presented another trapezoid that was not right and asked the subject to "apply that to a trapezoid without the special property." The subject did this successfully, but it was not clear why the length of the rectangle was $b_1 + 1/2\ (b_2 - b_1)$. The subject remarked, "Okay, it works out every time, but I don't see why. I do see why, I mean, I understand all the steps, it's just that logically looking at it, it doesn't make sense." I finally presented the idea that the rectangle's length was the average of the two bases, and this seemed satisfactory to the subject.

Subject 4. When Subject 4 was asked for the area of the trapezoid in Panel (a) of Figure 26, the subject drew two altitudes and tried to remember the formula. The subject mentioned the height of the figure, but did not remember the formula or ask for any specific information. I sketched a parallelogram and gave the base as 3 and asked for the area. The subject again drew an altitude but did not ask its length. It is quite possible that the subject assumed that the information given would be adequate to solve the problem. The altitude

constructed by the subject appeared to form an isosceles right triangle, and the subject attempted to use this but was unable to.

I drew another parallelogram and went through something like the Wertheimer-Resnick training procedure (see Resnick & Glaser, 1976) with the subject, who apparently understood the concept. Then I presented Panel (b) of Figure 26, with bases 3 and 5. The subject constructed an altitude, which I gave as 2 1/2. The subject found the area by decomposing the trapezoid into a rectangle and a triangle and adding the areas of the two parts. I worked out the formula with the subject, who then applied it correctly to the trapezoid in Panel (a), with the height given.

Subject 5. Subject 5 was asked to calculate the area of Panel (a), asked for the altitude, and said the area was the altitude times the average of the bases. The subject said that this was because the figure could be made into a rectangle. I asked about this, and the subject changed to a transformation into a parallelogram. The transformation used was the one involving a horizontal division and rotation of the top so that a long parallelogram is formed. Note that this makes the area (height over 2) X (sum of bases) rather than (height) X (average of bases). I surmise that the specific form of the transformation into a rectangle was not obvious to the subject, who thought of the alternative transformation.

I sketched a parallelogram, which the subject transformed into a rectangle by moving a triangle from one end to the other. Then I presented Panel (b) of Figure 26 and asked the subject to find a transformation involving an altitude midway between the ends of the two bases and specified that it would be at the midpoint of the right leg of the figure. When asked to go back to the formula, the subject noted that the length of the rectangle would be the average of the two bases.

We returned to Panel (a) of Figure 26, and I asked the subject for a transformation to a rectangle. The first transformation offered was the horizontal partition and rotation into a parallelogram with height of one-half that of the trapezoid, followed by moving a triangle from one end of the parallelogram to the other. When I asked for a transformation into a rectangle with the same height as the trapezoid, the subject produced the transformation involving altitudes midway between the ends of the bases.

Subject 6. When Subject 6 was given Panel (a), the subject asked for the altitude, then calculated the area using the formula. When I asked for a proof, the subject divided the trapezoid into a rectangle and two triangles, but was uncertain of how to find the lengths of the parts of the lower base forming the bases of the triangles. The subject realized that if one of these was known, the problem would be solvable.

I presented a parallelogram and asked for the formula for area and a proof. The subject gave Wertheimer's transformation. I then returned to Panel (a) of Figure 26 and asked for a transformation into a rectangle. The subject tried the transformation shown in Figure 27, apparently analogous to Wertheimer's transformation of the parallelogram. This probably was a case of trying to work forward, applying a known transformation to the situation rather than selecting a transformation based on a plan-based pattern related to the goal.

I presented Panel (b) of Figure 26 and asked the subject for the area. The subject proposed transforming the figure into a rectangle. The subject's first suggestion involved extending the top so it was the same length as the bottom and dropping a perpendicular. The subject realized that this rectangle had greater area than the trapezoid. Removing the triangle from the trapezoid was also considered, but the subject was apparently not making headway. I suggested that the subject might consider working with just part of the triangle, and then I

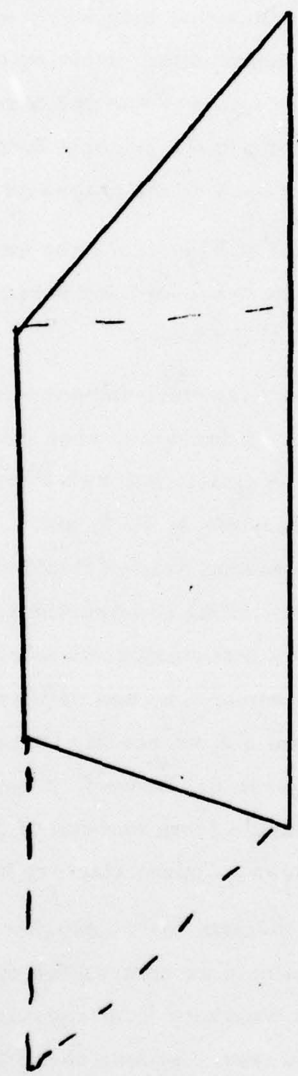


Figure 27. Transformation attempted by Subject 6 on Problem 9.

proposed that the rectangle should have the same altitude as the trapezoid and asked how long its length would be. The subject indicated it would be the sum of the bases divided by two. Following a suggestion to show where a line that long would end in the diagram, the subject generated the construction involving bisection of the right leg of the trapezoid and an argument that the triangle removed by that construction is congruent to the triangle formed by the construction and the extended upper base of the trapezoid.

Returning to Panel (a) of Figure 26, the subject applied the transformation to both ends of the trapezoid and noted that the resulting length is one-half the sum of the bases.

Discussion. The most frequent spontaneous construction was to divide the figure into forms for which area could be calculated easily, an action that seems consistent with Perdix's general procedure. This was done by Subjects 2, 3, 4, and 6, with Subject 3 performing the additional interesting transformation of concatenating the two triangles formed by the initial constructions. Subjects 2 and 5 generated or remembered a transformation into a parallelogram involving a horizontal partition. The use of Wertheimer's transformation of the parallelogram did not readily transfer to the trapezoid problem in the way he suggested. Instead, it suggested constructions involving removal of a triangle from one end of the trapezoid that are not productive and were seen as unsatisfactory by the subjects.

By and large, the schemata for rectangles and triangles appear to have been the salient guiding structures for these constructions, except in the cases where Wertheimer's transformation suggested use of parallel lines. However, in these cases the constructions were probably generated by local factors produced by my presenting the parallelogram problem in the situation.

Summary

In this section, the main characteristics of the protocols are summarized.

Problems 1 and 2

Students' work on Problems 1 and 2 was consistent with the idea that they had a pattern of parallel lines intersected by a transversal. A construction used by several students completed the pattern by extending oblique lines so they intersected with both parallels. This required using the sum of angles in a triangle to solve the problem. Another construction, used especially after its use in another problem, involved constructing an additional parallel line, from which the problems could be solved using just relations among angles with parallel sides. The results show that alternative forms of pattern completion can occur. The more salient idea for most students was completion of a transversal so that it would intersect both parallel lines. The idea of adding the third parallel line in the middle of the diagram was not used by some subjects, and for Subject 2 it seemed to have been made more salient by experiencing it in another problem.

These solutions displayed a variety of sequences regarding goals and patterns. Frequently the construction appeared to be motivated simply by having part of a pattern that seemed to be relevant, thus constituting a working-forward sequence. Other solutions seemed to involve rather complete knowledge about the way in which the problem would be solved once the pattern was completed. A possibility suggested by the protocols is that more experienced problem solvers can retrieve more differentiated patterns of potential action in the problem situation, thus permitting more complete anticipation of a problem solution.

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The processes seen here probably represent relatively global spatial-semantic processes that are used to produce patterns known to have relevant general features, with detailed formal use of those features worked out subsequently. An interesting case of that was Subject 2's performance on Problem 2, where the pattern produced a pair of adjacent angles, and this led to the use of algebra in formulating the details of the solution. Another use of spatial processing was evidenced by Subject 5 on Problem 2. The subject had a conjecture that three quantities sum to a constant, but noticed by mentally straightening a pair of segments that all three quantities increased together, thus disconfirming the conjecture.

Problems 3 and 4

These problems used a pattern with two triangles that share a side. The shared side is added to the diagram as a construction. In Problem 4, the construction involves connecting two points already designated in the diagram. This was solved in a straightforward way, apparently to complete the pattern, with details of the proof worked out later. This appears quite consistent with a top-down planning model. Problem 3 had an additional complexity, since the construction used only one point that was designated in the diagram and thus had to be specified with another property. The construction and its specification were apparently chosen simultaneously by two subjects, consistent with the idea that there are construction theorems that specify the nature of permitted constructions (Gerlernter, 1963). Three subjects appeared to make the construction in two stages, with the pattern completed by a line and the specification made separately. Of these, one subject's retrospection suggested that the specification was chosen from permissive theorems, and the other two apparently considered the goal of having appropriate congruent components for the higher goal of proving the triangles congruent.

In all cases, the choice of a construction was probably the result of processing in the semantic model, completing the pattern of triangles with a shared side, guided by the knowledge that this pattern was a prerequisite for a plan for constructing a formal solution of the problem. The semantic processing again appears to occur at a fairly global level, but has consequences involving features at a lower level that are used in syntactic processing.

Problems 5, 6, and 7

The patterns involved in solving these problems all involved triangles with varying additional constraints. In Problem 6, the plan used by Subjects 2 and 3 is based on the triangle inequality. These subjects added a diagnonal to the quadrilateral, thus forming a triangle whose third side could be bounded by sums and differences of sides of the quadrilateral, based on the triangle inequality. Then these bounds were considered in deducing bounds for the fourth side of the quadrilateral. Subject 2 noticed that the plan could be applied in two ways and checked the outcome of using the alternative diagonal. This probably resulted from scanning in the diagram, although a formal system could also have to select one of the two diagonals for the construction. Subject 3 engaged in some definite spatial processing in considering whether the result, obtained with a convex quadrilateral in the diagram, would also hold if the quadrilateral were concave. Note that the use of a construction is optional in this problem; a good solution can be obtained (and was by one subject, partially by another) by direct spatial analysis of the problem. The range of solutions is consistent with the idea that a construction is generated if the subject has a plan associated with the problem goal--in this case, triangle inequality with the goal of finding bounds on a side of a figure.

In Problem 7, the plan used (or tried) by all subjects was the Pythagorean theorem, which requires a right triangle. The hard

part of this problem was apparently finding the major problem goal. This requires sophisticated spatial processing; the two subjects who succeeded drew good diagrams representing the three-dimensional object clearly. Having identified the goal clearly, it was apparently straightforward to determine that a right triangle having that diagonal as a side could be constructed, using a diagonal of a surface as one of its legs and an edge as the other leg. Some fairly powerful spatial processing must be involved in this, but it seems plausibly a case of pattern completion guided by the plan of using the Pythagorean theorem and the consequent goal of constructing an appropriate right triangle. Note that for Subject 6, the goal of using right triangles was also salient, but the spatial representation available did not lead to finding the right triangle that was needed.

Performance on Problem 5 can be interpreted as the result of lacking an appropriate plan. Subject 2 apparently remembered some features of the proof of the inequality, including the exterior angle and an isosceles triangle. However, in constructing the triangle the subject used the wrong parts of the triangle and was unable to get a proof. Other subjects appeared to be casting about for a helpful construction, considering the bisector of an angle, a median, or an altitude. Especially in the case of Subject 4, this seemed to be chosen on the basis of analogy with the problem of proving congruence of the base angles of an isosceles triangle. The triangle inequality was mentioned by three subjects, and the sum of angles in a triangle was mentioned by one subject. It may be that these subjects were searching for a relevant formal proposition that might be used in a plan to solve the problem.

Problem 8

This problem requires use of a pattern of a central angle for the arc of a circle. The case analysis required for generating a proof was not produced spontaneously by these subjects, and only two of them

seemed to remember the cases clearly after instruction had been given. Subject 2 was able to solve the problem in Case 2, including generating the diameter as a construction after making the central angle. This solution depended on using algebra, showing an interesting use of a formal system to keep track of the numerous components that had to be related. Subject 3's solution to the problem depended on a suggestion to try Case 1, where the subject included the central angle easily. In Case 2, Subject 3 was able to apply the solution of Case 1 quite directly, first constructing the diameter and then working out the additive relationships informally.

Subjects 3, 5, and 6 all completed inscribed triangles by adding chords at least once, and for Subject 5 this was a serious distraction. Subject 3 rejected the idea when the only formal property that was recalled involved congruence of chords which seemed irrelevant to the problem goal. Subject 5 also considered bisecting the arc, a sensible attempt to relate one-half of its measure to the inscribed angle.

The use of the central angle in the auxiliary problem apparently made the pattern with the central angle more salient for Subjects 5 and 6, who were able to prove Case 1 of the theorem after solving that problem.

Problem 9

This problem provided further illustrations of pattern-based constructions, but the subjects also showed several examples of interesting interaction between formal and spatial reasoning.

Three of the five subjects began by constructing altitudes that partitioned the given trapezoid into a rectangle and two triangles. This seems a straightforward example of a plan-based construction. The goal was to find area; formulas were known for rectangles and triangles, but prerequisites for those are presence of rectangles or

triangles. The constructions produced these. To use the constructed figures, the problem solver must identify their dimensions. (This requirement is probably identified in the formalism.) Subjects 2 and 5 were unable to identify the bases of the triangles; note that Subject 2 was trying to do this with general notation (b_1 and b_2) rather than with numerical values. Subject 3 solved the problem by an ingenious transformation in which the two triangles were concatenated to form a single triangle with known height and total base of $b_2 - b_1$.

The protocols included an interesting case of generalization, apparently involving interaction between spatial and formal properties. It was especially clear for Subject 2, although there was at least a recognition of the similarity by Subject 3 on the same point. In the right trapezoid, only one altitude is needed, forming a rectangle and a single triangle. In this situation, the length of the triangle's base is more easily seen as the difference between the two bases of the trapezoid. (Note that Subject 4 also identified the difference in this case.) Having found a formula including reference to the difference (presumably cued by spatial features), Subject 2 then transferred that formal idea to the general case, with help from algebra (the bases of the two triangles were labeled x and y). If we interpret the solution of the problem for the right trapezoid as the building of a production with the difference between the bases as one of its components, then the solution of the general case seems to involve generalizing the conditions for applying the production, including some procedures for combining separate components of the situation.

The protocols also contained illustrations of using transformations that had been learned previously, sometimes with interesting modifications in response to new constraints. A transformation that was apparently remembered and applied directly by Subject 2 and 5 involved "slicing" the trapezoid horizontally and rotating the upper half to form a parallelogram. More interesting instances occurred

after Wertheimer's transformation of the parallelogram was presented in the situation. Subjects 2 and 3 both applied this transformation to the trapezoid by removing a triangle from one end of the trapezoid. The construction used was to draw a line from the upper left corner parallel to the right leg of the trapezoid. Note that this modifies one of the properties of the transformation as it occurs in the parallelogram problem. There the construction involves an altitude, which produces a figure in which the vertical sides are parallel (and, as it happens, perpendicular to the bases). When the parallel line is constructed, there is no way to move the triangle to form a parallelogram, although the parallelogram pattern may have served to motivate the construction initially.

Subject 6's response after the presentation of Wertheimer's transformation preserved the constraint of a perpendicular line. When the triangle was moved to the other side, it was rotated rather than being simply translated as it is in the parallelogram. Further, it does not fit unless there is a transformation made on the left side to obtain a leg perpendicular to the bases. The subject's analysis of the result of the transformation was rather weak; the subject noted the parallel lines in the diagram and inferred that there would be a parallelogram (apparently forgetting that the triangle from the right side was being removed). Note that Subject 5 also considered constructing an altitude and translating the triangle thus formed to the other end of the trapezoid when I asked for a transformation into a rectangle. This is a sensible construction to attempt since it creates some of the features of a rectangle (the right angles at the right end of the figure) and corresponds to the first steps of Wertheimer's transformation, so is "the same" in a noticeable sense. However, it cannot be completed, as Subject 5 and 6 both recognized.

The request to transform a right trapezoid into a rectangle led three subjects to add a triangular region to the trapezoid, and Subject 6

also considered removing a triangle. The constraint to have the same area led Subjects 3 and 5 to make the construction midway between the endpoints of the bases (recall Subject 3's remark that it would be "a little fatter"). For Subject 6, an additional hint was needed, involving the suggestion to use part of the triangle and keep the altitude of the trapezoid.

The generalization of the transformation of the right trapezoid to the general case was accomplished easily by Subjects 3 and 6, although not by Subject 5. For this subject, an additional constraint was given as a hint to keep the rectangle the same height as the trapezoid. (Recall that Subject 5 first applied the transformation of dividing the trapezoid horizontally and rotating the top portion.)

It seems quite clear that for finding area, plans involving decomposition were more intuitively salient for these subjects than plans involving area-preserving transpositions into known forms. On the other hand, the decomposition of a trapezoid provides little or no intuition for the formula $A = 1/2h\,(b_1 + b_2)$. It is interesting to consider a sequence of spatial transformations that bear some relationship to a set of algebraic manipulations that take the decomposition into the rectangle. Figure 28 shows one such sequence, starting with the addition of length to the shorter base until a rectangle has been formed with the same area as the trapezoid. The algebraic transformation just factor is h, corresponding to the constraint of preserving the height. The second transformation involves a kind of recentering or refocusing, where the length of the base resulting from the first transformation is seen as the average of the two initial bases. Algebraically, an expression is simplified. It may be that this recentering is improbable; at least in Subject 3's response to the idea, it was not clear that there was a strong intuitive basis for the idea of an average length.

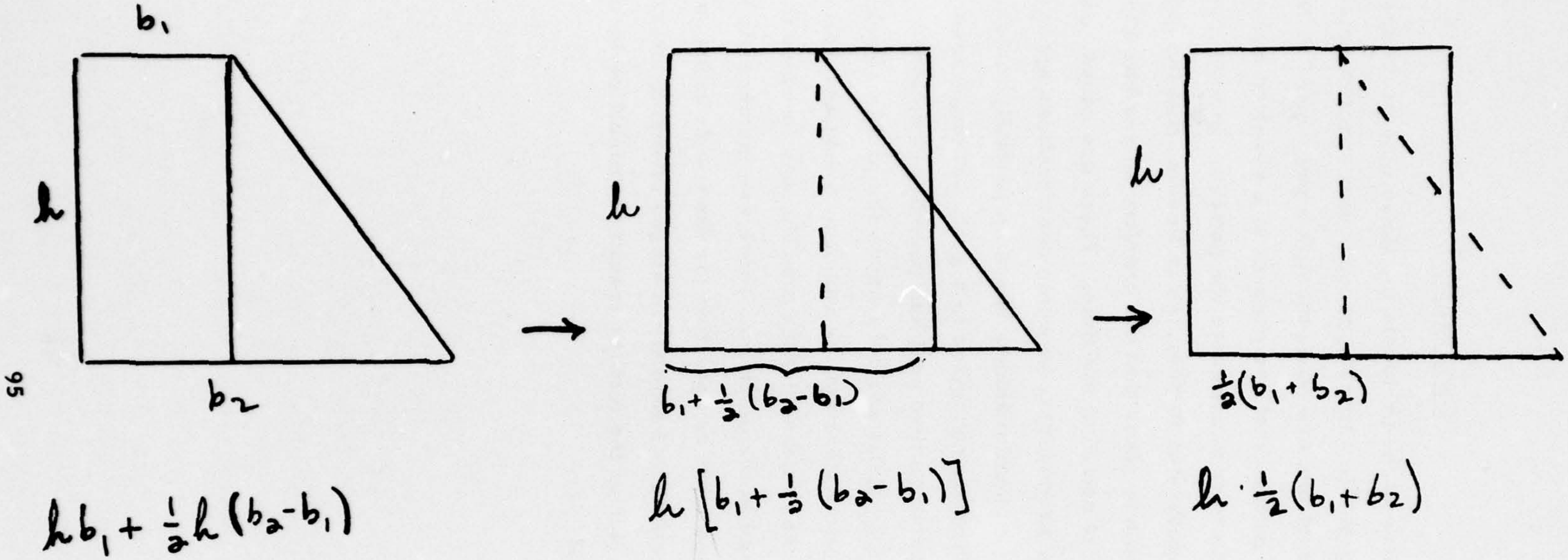

Figure 28. Geometric transformations corresponding to algebraic steps in derivation of the formula for area of a trapezoid.

Conclusions

Most of the data in the protocols for these nine problems seem quite clearly interpretable using the general idea, programmed in Perdix, that constructions are made through a process of matching some features of a pattern that are present in a situation and adding missing features in order to complete the pattern. In most cases, it is reasonable to hypothesize further, as in Perdix, that the patterns are chosen in relation to plans that the problem solver has associated with the problem goal active at the time. There are cases, contrary to Perdix's specific procedures, in which constructions appear to be added in a working-forward manner, based on partially matched patterns that are not directly connected to a goal, but these seem to be considerably less common than goal and plan-based constructions. While constructions typically are relevant to the problem goal in a general way, they usually are not the result of a detailed plan in which the problem solver has worked out the specific way in which the construction will be used in the solution. There are instances in which geometric properties must be specified for the construction, and the construction is specified in a number of stages involving consideration of the possible use of the further specified properties in achieving the problem goal.

References

Chase, W. G., & Simon, H. A. Perception in chess. Cognitive Psychology, 1973, 4, 55-81.

Gelernter, M. Realization of a geometry-theorem proving machine. In E. A. Feigenbaum & J. (Eds.), Computers and thought. New York: McGraw-Hill, 1963.

Greeno, J. G. A study of problem solving. In R. Glaser (Ed), Advances in instructional psychology (Vol. 1). Hillsdale, N. J.: Lawrence Erlbaum Associates, 1978.

Greeno, J. G., Magone, M. E., & Chaiklin, S. Theory of set and constructions in problem solving. Pittsburgh: University of Pittsburgh, Learning Research and Development Center, 1979. (LRDC Publication 1979/9)

Reitman, J. S. Skilled perception in Go: Deducing memory structures from inter-response times. Cognitive Psychology, 1976, 8, 336-356.

Resnick, L. B., & Glaser, R. Problem solving and intelligence. In L. B. Resnick (Ed.), The nature of intelligence. Hillsdale, N. J.: Lawrence Erlbaum Associates, 1976.

Sacerdoti, E. D. A structure for plans and behavior. New York: Elsevier, 1977.

Wertheimer, M. Productive thinking. New York: Harper & Row, 1959.

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